Show all of your work and all of your answers in your bluebook. When you are finished, hand in this question sheet together with your bluebook. Good luck.

(1) (10pts) Suppose $A \in M_{m \times n}(\mathbb{F})$ and $B \in M_{n \times p}(\mathbb{F})$. Prove that $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$.

(2) (12pts) $A$ is a $3 \times 3$ matrix and $b$ is a $3 \times 1$ matrix such that the system $Ax = b$ is solved by \[
\begin{bmatrix}
1 \\
2 \\
-2
\end{bmatrix}
\text{ and } \begin{bmatrix}
-2 \\
1 \\
0
\end{bmatrix}, \text{ but not by } \begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix}.
\] Determine which of the following statements (if any) are certainly true based on the given information, which (if any) are certainly false, and which (if any) could be true or false. (Justify your answers.)
(a) $Ax = 0$ has infinitely many solutions.
(b) $-A \begin{bmatrix}
5 \\
0 \\
-2
\end{bmatrix} = b$
(c) $\begin{bmatrix}
3 \\
2 \\
0
\end{bmatrix}$ fails to solve the system $Ax = b$.
(d) $\begin{bmatrix}
1 \\
3 \\
0
\end{bmatrix}$ fails to solve the system $Ax = b$.

(3) (12pts) (a) Test $A = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 2 \\
2 & 0 & 3
\end{bmatrix}$ for diagonalizability. If $A$ is diagonalizable, find a diagonal $D$ and an invertible $Q$ such that $A = QDQ^{-1}$.
(b) Test $A = \begin{bmatrix}
-3 & 1 & -1 \\
-10 & 4 & -1 \\
6 & -1 & 4
\end{bmatrix}$ for diagonalizability. If $A$ is diagonalizable, find a diagonal $D$ and an invertible $Q$ such that $A = QDQ^{-1}$.
(Show your work.)

(4) (6pts) Suppose $A, M \in M_{n \times n}(\mathbb{F})$ satisfy $2M^6 + M^3 + A^2 = O$.
(a) If $A$ is invertible does it follow that $M$ is invertible also?
(b) If $A$ is singular does it follow that $M$ is singular also?
(Justify your answers.)

(5) (8pts) Give a concise proof that $\{\sin x, \sin 2x, \sin 3x, \ldots\}$ is a linearly independent subset of $C^\infty(\mathbb{R}, \mathbb{R})$.

(6) (8pts) Let $\beta$ and $\beta'$ be the ordered bases $\beta = \{(0,0,1), (1,0,0), (1,1,1)\}$ and $\beta' = \{(1,1,0), (1,0,1), (0,1,1)\}$ for $\mathbb{R}^3$. Find the change of coordinate matrix from $\beta'$ to $\beta$.
(Show your work.)
(7) (8pts) Suppose $T$ is a linear operator on an $n$-dimensional vector space $V$, and $V$ is a $T$-cyclic subspace of itself. Let $W$ be the subspace of $L(V)$ that consists of the operators on $V$ that commute with $T$ (i.e., $W = \{ U \in L(V) : TU = UT \}$). Can you determine bounds on $\dim(W)$ (as a function of $n$)? Can $\dim(W)$ be determined? (Justify your answers.)

(8) (8pts) Compute \[
\begin{bmatrix}
1 & -1 & 0 \\
3 & -2 & 1 \\
-1 & 1 & 1
\end{bmatrix}^{15}.
\]
(Show your work.)

(9) (8pts) For each $A \in M_{3 \times 3}(\mathbb{F})$ define the operator $T_A : M_{3 \times 3}(\mathbb{F}) \to M_{3 \times 3}(\mathbb{F})$ by $T_A(M) = AM$. Does there exist an $A$ and an $M$ such that the $T_A$-cyclic subspace generated by $M$ is all of $M_{3 \times 3}(\mathbb{F})$? (Justify your answer.)

(10) (8pts) Let $W = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 + 2x_3 + 4x_5 = 0 \text{ and } 2x_1 - x_2 + x_3 - x_5 = 0\}$.
(a) Show that $S = \{(2, 3, -1, 0, 0), (0, -3, -2, 1, 1)\}$ is a linearly independent subset of $W$.
(b) Extend $S$ to a basis of $W$.
(Show your work.)

(11) (12pts) Consider each of the following statements carefully and determine which are true and which are false. Remember that a general claim is false if there are any exceptions.
(a) $T : P(\mathbb{R}) \to P(\mathbb{R})$ defined by $T(f) = \int_0^x p(t) dt$ is invertible.
(b) The product of elementary matrices is an elementary matrix.
(c) $P_3(\mathbb{R})$ is isomorphic to $\mathbb{R}^3$.
(d) If $A \in M_{n \times n}(\mathbb{F})$ then the span of the rows of $A$ equals the span of the columns of $A$.
(e) If $A \in M_{n \times n}(\mathbb{F})$ and $A^3$ is singular, then $A$ itself must be singular.
(f) If $v_1, \ldots, v_k$ are eigenvectors of linear operator $T$, then every non-zero vector in $\text{span}\{v_1, \ldots, v_k\}$ is an eigenvector of $T$.
(g) If $A \in M_{n \times n}(\mathbb{F})$ has no eigenvalues of algebraic multiplicity strictly greater than 1, then the trace of $A$ equals the sum of the eigenvalues of $A$.
(h) If $A \in M_{n \times n}(\mathbb{F})$ and $A + A$ is singular, then $A$ itself must be singular.
(i) If $v_1, \ldots, v_k$ are eigenvectors of linear operator $T$, then $\{v_1, \ldots, v_k\}$ is linearly independent.
(j) $L(\mathbb{C}^2, \mathbb{C}^3)$ is isomorphic to $\mathbb{C}^6$.
(k) $\det : M_{3 \times 3}(\mathbb{R}) \to \mathbb{R}$ is linear.
(l) If $T : M_{n \times n}(\mathbb{R}) \to M_{n \times n}(\mathbb{R})$ is linear, then there exists $A \in M_{n \times n}(\mathbb{R})$ such that $T(B) = AB$ for all $B \in M_{n \times n}(\mathbb{R})$. 