MA-113 Constrained optimization exploratory activity V0.2: 2020-05-26
Part 1: Preparation

Introduction:
This activity will guide you through a graphical exploration of the method of Lagrange multipliers for solving constrained optimization problems. The central ideas will be illustrated with an example similar to the following exercise.

> (*) Find the maximum value of
>
> $$f(x, y, z) = x + y - \frac{z}{3}$$
>
> subject to the pair of constraints
>
> $$\begin{cases} x + 2y + 3z = 2 \\ x^2 + y^2 + z^2 = 1 \end{cases}.$$

The preparatory questions below are to be completed prior to the in-class portion of the activity.

Preparatory questions:

(1) Describe the family of level surfaces of the objective function f.

(2) Does the objective function f obtain a maximum value in xyz-space? Give an explanation which relates your answer to the family of level surfaces of f.

(3) Describe some subsets of xyz-space on which f does attain a maximum value. (Try to produce a variety of examples with qualitative differences.)

(4) Describe some subsets of xyz-space on which f does not attain a maximum value. (Again try to produce a variety of examples with qualitative differences.)

(5) Suppose $\boldsymbol{r}(t) = (x(t), y(t), z(t))$, $a < t < b$ is a regular parameterization of a space curve. Use the chain rule to determine the rate of change of f (with respect to t) as f is evaluated along the curve. (In other words, compute $\frac{d}{dt}\left(f(\boldsymbol{r}(t))\right)$.) Express your answer explicitly in terms of the gradient of f and the velocity of $\boldsymbol{r}$.

(6) From the origin, in which direction should you go to increase the value of f as rapidly as possible? Explain.

(7) Describe the set of *all* directions in which one can move from the origin so that f will increase. Explain. (Feel free to discuss the subtleties of slightly different interpretations of this exercise with your instructor.)

(8) In questions (6) and (7), if the origin is replaced by an arbitrary space point P do the answers change?

(9) Describe the solution set of the first constraint equation $x + 2y + 3z = 2$ from sample problem (*) above.

(10) Describe the solution set of the second constraint equation $x^2+y^2+z^2 = 1$.

(11) In general what types of sets can occur as the intersection of a plane and a sphere? Comment on the problem of optimizing a continuous function in each case.

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Part 2: Graphical Exploration / CAS Computation

Prerequisites: (Essential) Understanding of level surfaces, gradients, and directional derivatives. (Desirable) Experience optimizing a function of two variables subject to a constraint condition satisfied by a curve in the function's domain.
Resource URL: *Temporary*
`http://faculty.cooper.edu/smyth/threejs/calculus/lagrangeMultipliers.htm`

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Guide / discussion questions:

(1) Open the webpage at the URL given above. Expand the Choose objects folder from the control panel in the upper right hand corner. Select the Constraint 1 (plane) and Constraint 2 (sphere) checkboxes to add the solutions of the constraint equations to the main canvas. The graphic can be repositioned and reoriented using either a mouse (or touchscreen), or the Rotational controls. In this example, which type of intersection is formed by the plane and sphere?

(2) Select Constraint curve to add the circle of points simultaneously satisfying both constraint conditions to the graphic. How could you have determined that $x + 2y + 3z = 2$ and $x^2 + y^2 + z^2 = 1$ intersect in a circle (rather than

just a single point, or the empty set) without using a computer?

(3) Select Test point to add a distinguished point to the constraint circle. You can move this marked point to any position on the constraint circle using the Test point controls. At any given fixed position on the constraint curve, how many directions are possible for motion which does not leave the constraint curve? (More precisely, suppose $\gamma : (a, b) \to \mathbb{R}^3$ is a regular parameterization for some portion of the constraint curve including point P, and $t_0 \in (a, b)$ is a parameter value such that $\gamma(t_0) = P$. How many possible directions could the velocity vector $\gamma'(t_0)$ have?)

(4) Select the two Tangent to constraint checkboxes to add visual representations of the two "constraint sensitive" directions at the marked test point. What information about the objective function (the function we want to maximize subject to the constraints) would facilitate a determination of whether it will increase or decrease as the test point is moved along the constraint curve?

(5) Select Objective function gradient. What can you say about the directional derivative of f at a point P in a direction which makes an acute angle with $\nabla f(P)$? What can you say about how $f(P)$ will change as long as the direction in which P is moved continues to form an acute angle with $\nabla f(P)$?

(6) Select Objective function monitor. Using the Test point controls move the test point (point "P") around the constraint circle. View the left pane graphical representation of how the objective function values change (and/or the textual display on the left side of the top banner) to confirm your answers above. State a relationship between $\nabla f(P)$ and the tangent line to the constraint curve at P that *precludes* the possibility of f attaining an extreme value at P subject to the constraints. Recast your observation as a *necessary* condition for f to attain an extreme value at P (subject to the constraints). Is this condition, in general, a *sufficient* condition?

(7) Define $g(x, y, z) = x + 2y + 3z - 2$ and $h(x, y, z) = x^2 + y^2 + z^2 - 1$ so that the constraint conditions can be written as $g = 0, h = 0$. In particular, the constraint surfaces are level surfaces of the functions g and h. Given P on the intersection of the two constraint surfaces, what's the relationship between $\nabla g(P)$ and the constraint plane, and what's the relationship between $\nabla h(P)$

and the constraint sphere?

(8) If f attains an extreme value subject to the constraints $g = 0, h = 0$ at point P, must $\nabla f(P)$ be parallel to $\nabla g(P)$ and/or $\nabla h(P)$?

(9) Select Normal to plane, Normal to sphere, and Span of constraint normals. Recast your necessary condition for f to attain a constrained extreme value at P (from step (6)) as a requirement on $\nabla f(P)$ in terms of $\nabla g(P)$ and $\nabla h(P)$. Formulate this condition in both geometric and linear algebraic terms.

(10) Record the coordinates (as precisely as you can) for the point where f achieves its maximum subject to the constraints. What is the constrained maximum value of f?

(11) Select Obj fcn level surf's to display a few select level surfaces of the objective function f. What relationship holds between level surfaces of f and points where constrained extreme values of f occur in this example? (For this step you're advised to unselect objects to reduce clutter, and unselect the Perspective camera to switch to an orthographic view.) Can you generalize this relationship? (Beware. This is a bit tricky.)

CAS Symbolic Computation

(1) Open MATLAB and prime the Symbolic Toolbox with a symbol declaration.

```
>> syms x y z lambda mu
```

(2) Define the objective function, and the functions appearing in the constraint equations.

```
>> f = x + y - z/3
>> g = x + 2*y + 3*z - 2
>> h = x^2 + y^2 + z^2 - 1
```

(3) Define the Lagrangian for our example problem.

```
>> lagrangian = f - lambda*g - mu*h
```

(4) Encode the necessary condition for constrained extrema as a single vector equation. (Don't confuse the double equal sign `==` [equality] with the single equal sign `=` [assignment].)

```
>> eqn = ( gradient(lagrangian, [x y z lambda mu]) == [0; 0; 0; 0; 0] )
```

(5) Solve the system.

```
>> candidates = solve( eqn, [x y z lambda mu] )
```

(6) Extract the spacial coordinates from the solutions (and peek at numerical approximations).

```
>> [ candidates.x candidates.y candidates.z ]
>> vpa( ans )
```

(7) Evaluate f at the candidate points.

```
>> subs( f, [x y z], [candidates.x(1) candidates.y(1) candidates.z(1)] )
>> vpa( ans )
>> subs( f, [x y z], [candidates.x(2) candidates.y(2) candidates.z(2)] )
>> vpa( ans )
```

(8) Where is the constrained maximum of f achieved and what is its value? (Is your answer here consistent with your answer to question (10) above?)

MA-113 Constrained optimization exploratory activity
Part 3: Analytical solution

V0.2: 2020-05-26

(*) Find the maximum value of

$$f(x, y, z) = x + y - \frac{z}{3}$$

subject to the pair of constraints

$$\begin{cases} x + 2y + 3z = 2 \\ x^2 + y^2 + z^2 = 1 \end{cases}.$$

(1) Solve the constrained optimization problem (*) with paper and pencil using the method of Lagrange multipliers. Organize your work as a clear, linear presentation with explanatory notes for each step.

(2) What if the constraint in (*) is revised to allow all points of the plane $x + 2y + 3z = 2$ on *or inside* the sphere $x^2 + y^2 + z^2 = 1$?

(3) What if the planar constraint is dropped? Can you then solve the problem without calculus?

(4) Re-solve (*) without Lagrange multipliers by finding an explicit parameterization for the constraint curve.

(5) Which of the two solution methods for (*) (Lagrange multipliers vs. explicit parameterization of the constraint curve) do you think is more flexible in general? Why?

(6) Find the maximum and minimum values of $2x^2 + 2y - z$ subject to the constraints $z = 2x^2 + 2y^2$ and $4y^2 + z^2 = 4$.

(7) Can you identify any feature(s) of optimization problem (6) not in evidence in (*)?