Linear Algebra Review / Primer

- Fields
- Vector Spaces
- Linear Transformations
- Inner Products
- Dual Spaces

Fields

Definition

A *field* $\mathbb{F}$ is a number system where addition and multiplication satisfy:

- $a + b = b + a$
- $(a + b) + c = a + (b + c)$
- $\exists 0 \in \mathbb{F}$ such that $\forall a \in \mathbb{F}(0 + a = a)$
- $\forall a \in \mathbb{F}(\exists - a \in \mathbb{F}$ such that $a + (-a) = 0)$
- $ab = ba$
- $(ab)c = a(bc)$
- $\exists 1 \in \mathbb{F}$ such that $\forall a \in \mathbb{F}(1a = a)$
- $\forall a \in \mathbb{F} \setminus \{0\}(\exists a^{-1} \in \mathbb{F}$ such that $aa^{-1} = 1)$
- $a(b + c) = ab + ac$
- $0 \neq 1$

Fields

Examples

- $\mathbb{Q}$ (the rationals)
- $\mathbb{R}$ (the reals ... our favorite field!!!)
- $\mathbb{C}$ (the complex numbers)
- $\mathbb{Z}_p$ (integers mod p, where p is prime)

Close... but no cigar

- $\mathbb{Z}$ (the integers) $\exists$ nonzero integers w/o multiplicative inverses in $\mathbb{Z}$
- $\mathbb{H}$ (the quaternions) multiplication fails to be commutative
- $\mathbb{Z}_n$ (integers mod n, where n is composite)

Vector Spaces

Definition

A *vector space* V over field $\mathbb{F}$ is a set of "vectors" that can be added to each other, and scaled by elements of $\mathbb{F}$ so that:

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\exists \mathbf{0} \in V$ such that $\forall \mathbf{u} \in V(\mathbf{u} + \mathbf{0} = \mathbf{u})$
- $\forall \mathbf{u} \in V(\exists - \mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0})$
- $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
- $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
- $(ab)\mathbf{u} = a(b\mathbf{u})$
- $1\mathbf{u} = \mathbf{u}$

Vector Spaces

Quintessential finite dimensional example

$\mathbb{R}^n$ is a vector space over $\mathbb{R}$ with addition and scaling defined componentwise

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

$$a(x_1, x_2, \ldots, x_n) = (ax_1, ax_2, \ldots, ax_n)$$

An infinite dimensional example

$C^n((0,1);\mathbb{R})$, the space of n times continuously differentiable real-valued functions on the interval $(0,1)$, is a vector space over $\mathbb{R}$ with addition of functions (in this context a.k.a. "vectors") and scaling performed pointwise.

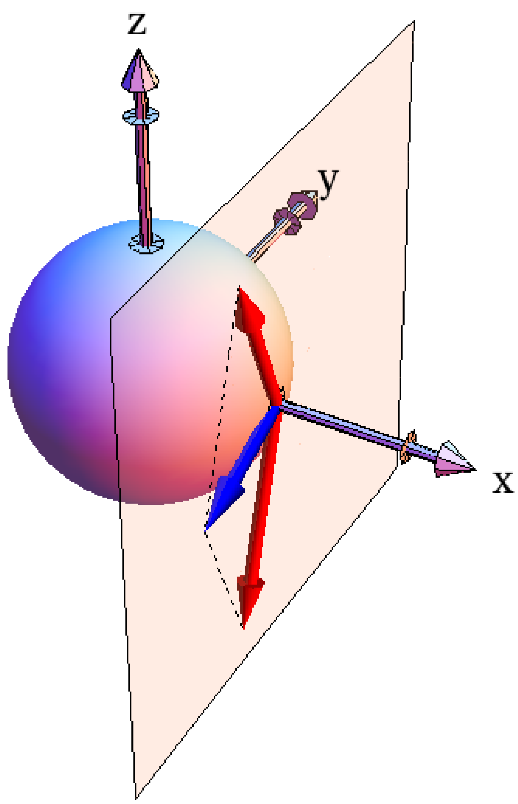
Vector Spaces

A relevant example

$T_{(1,0,0)}S^2 = \{((1,0,0),(0,y,z)) : y, z \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$ with addition and scaling defined by

$$((1,0,0),(0,y_1,z_1)) + ((1,0,0),(0,y_2,z_2)) = ((1,0,0),(0,y_1+y_2,z_1+z_2))$$

$$a((1,0,0),(0,y,z)) = ((1,0,0),(0,ay,az))$$

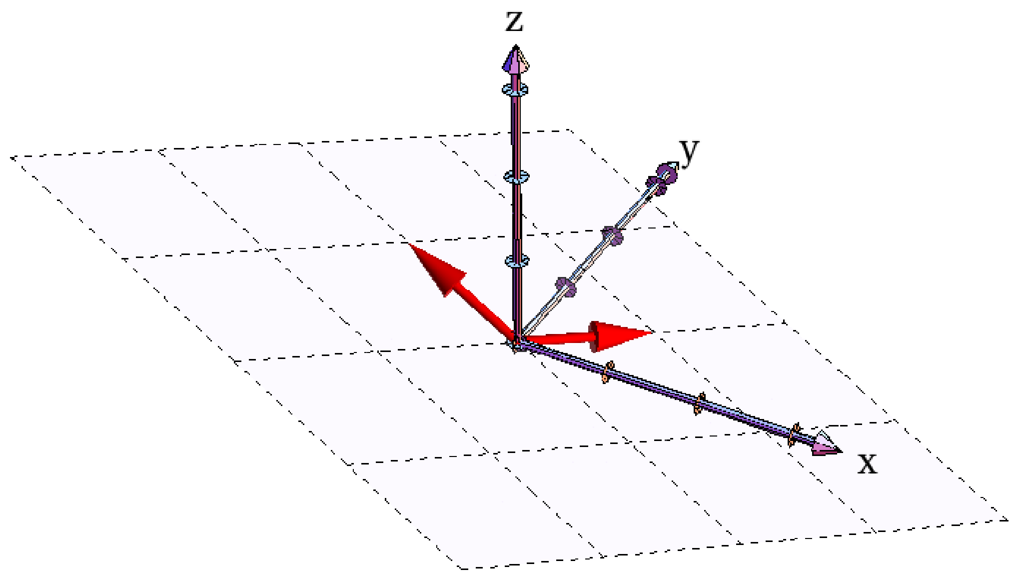


Vector Spaces

Definition

The *span* of $S \subset V$ is the set of all linear combinations of vectors in S.

$$span(S) = \{a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k : a_1, \ldots, a_k \in \mathbb{F}, \mathbf{u}_1, \ldots, \mathbf{u}_k \in S\}$$



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Definition

A set of vectors S in V is *linearly independent* if no vector in S is contained in the span of the other vectors in S.

Definition

A *basis* for V is a linearly independent set of vectors that spans V.

Example

$$\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\} \text{ is a basis for } \mathbb{R}^3.$$

Vector Spaces

Definition

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Example

Recall $T_{(1,0,0)}S^2 = \{((1,0,0),(0,y,z)) : y,z \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$ with addition and scaling defined by

$$((1,0,0),(0,y_1,z_1)) + ((1,0,0),(0,y_2,z_2)) = ((1,0,0),(0,y_1+y_2,z_1+z_2))$$

$$a((1,0,0),(0,y,z)) = ((1,0,0),(0,ay,az)).$$

$$\{\, ((1,0,0),(0,1,0)),\ ((1,0,0),(0,0,1))\,\} \text{ is a basis for } T_{(1,0,0)}S^2.$$

Vector Spaces

Definition

A *basis* for V is a linearly independent set of vectors that spans V.

Theorem

Any two bases for a given vector space V have the same cardinality.

This justifies the following definition.

Definition

The *dimension* of a vector space V is the number of vectors in any basis for V.

Examples

The dimension of $\mathbb{R}^n$ is n.
The dimension of $T_{(1,0,0)}S^2$ is 2.

Linear Transformations

Definition

Suppose V and W are vector spaces over a common field $\mathbb{F}$. Then $T : V \to W$ is *linear* if $\forall a, b \in \mathbb{F}$, $\forall \mathbf{u}, \mathbf{v} \in V$

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

Example

$\frac{d}{dx} : C^n((0,1);\mathbb{R}) \to C^{n-1}((0,1);\mathbb{R})$ is linear (for $n = 1, 2, \ldots$)

Example

Given an $m \times n$ matrix M (with real entries), the map $L_M : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$L_M(\mathbf{u}) = M\mathbf{u}$$

is linear.

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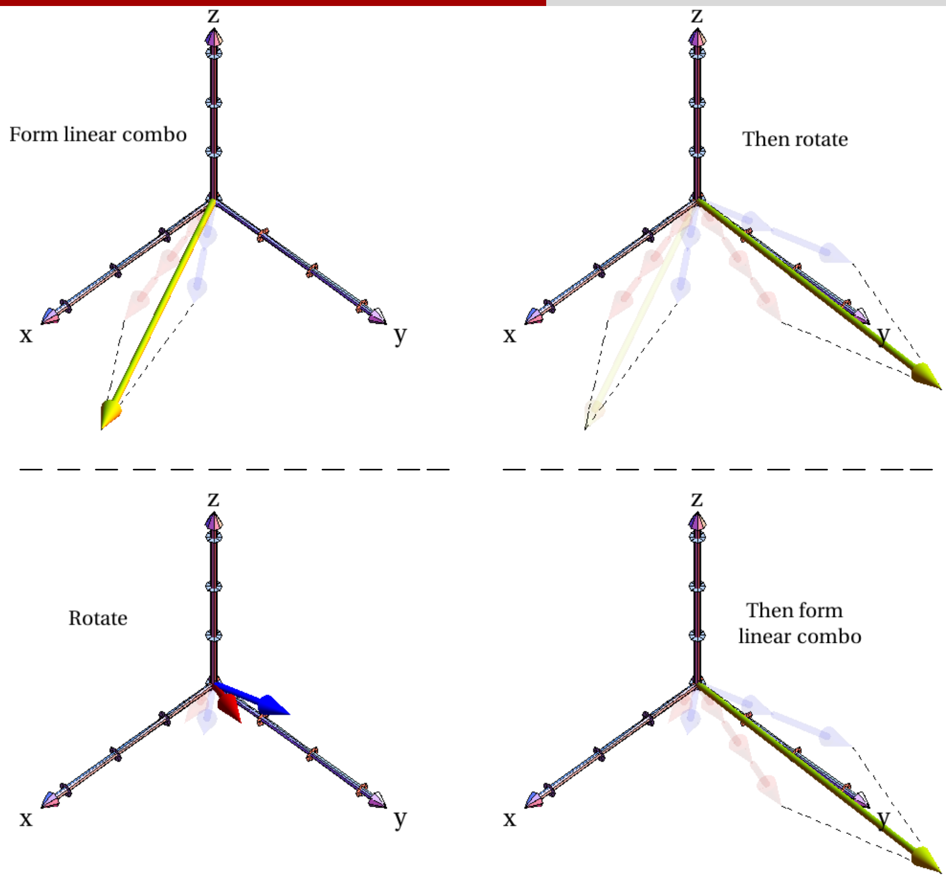
$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

Example

Any rotation about any axis through the origin of space is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.
Consider for example rotation by $\dfrac{\pi}{4}$ radians about the z-axis counterclockwise (as viewed from above). (Incidentally, this transformation is equal to L_R where $R = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.)



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Surprise

$f : \mathbb{R}^1 \to \mathbb{R}^1$ given by $f(x) = 3x + 5$ is *not* linear

Inner Products

Definition

An *inner product* on vector space V over $\mathbb{R}$ is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that:

- $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
- $\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle$
- $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ (with $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ iff $\mathbf{u} = \mathbf{0}$)

Prototypical Example

Ordinary dot product on $\mathbb{R}^n$

For $\mathbf{u} = (u_1, \ldots, u_n)$, $\mathbf{v} = (v_1, \ldots, v_n)$, the dot product $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i$ is an inner product.

Note that, if $\mathbf{u}$ and $\mathbf{v}$ are written as $n \times 1$ matrices (a.k.a. column vectors), then $\mathbf{u} \cdot \mathbf{v}$ may be written as $\mathbf{u}^\intercal \mathbf{v}$ or $\mathbf{u}^\intercal \mathcal{I} \mathbf{v}$.

Inner Products

A Useful Observation

If M is a positive definite matrix (i.e. symmetric with all positive eigenvalues), then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\intercal M \mathbf{v}$$

is an inner product.

For example $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^\intercal \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an inner product on $\mathbb{R}^2$.

Another example

$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ is an inner product on $C^0([0,1];\mathbb{R})$.

Inner Products

Euclidean norm

The ordinary Euclidean length (a.k.a norm) of vector $\mathbf{u} \in \mathbb{R}^n$ may computed using the ordinary dot product:

$$||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

A generalization

Any inner product on a vector space V may be used to define a notion of length for the vectors in V:

$$||\mathbf{u}|| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

Arclength

A key ingredient in the computation of arclength is the computation of length of velocity vectors:

$$L = \int_{t_0}^{t_1} ||\mathbf{v}(\tau)|| d\tau$$

Dual Spaces

Definition

Given a vector space V over $\mathbb{R}$, a linear transformation from V to $\mathbb{R}$ is called a *linear functional* (or *dual vector* or *covector*). The set of all dual vectors (endowed with its natural vector space structure) is called the *dual space*. The dual space is denoted V^*.

Example

If $V = \mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}^3$, then $dot_{\mathbf{w}} : V \to \mathbb{R}$ defined by $dot_{\mathbf{w}}(\mathbf{v}) = \mathbf{v} \cdot \mathbf{w}$ is a dual vector. Of course
$dot_{\mathbf{w}}(a\mathbf{u} + b\mathbf{v}) = (a\,\mathbf{u} + b\,\mathbf{v}) \cdot \mathbf{w} = a\,\mathbf{u} \cdot \mathbf{w} + b\,\mathbf{v} \cdot \mathbf{w} = a\,dot_{\mathbf{w}}(\mathbf{u}) + b\,dot_{\mathbf{w}}(\mathbf{v})$.
So $dot_{\mathbf{w}}$ is, as advertised, linear from V to $\mathbb{R}$.

Dual Spaces

Theorem

Suppose $\lambda \in (\mathbb{R}^3)^*$. Then there exists $\mathbf{w} \in \mathbb{R}^3$ such that $\lambda = dot_{\mathbf{w}}$.

Proof

Suppose λ is a dual vector on $\mathbb{R}^3$.
Let $\mathbf{w} = (\lambda(\hat{\mathbf{i}}), \lambda(\hat{\mathbf{j}}), \lambda(\hat{\mathbf{k}}))$.
Now suppose $\mathbf{u} = (u_1, u_2, u_3)$ is an arbitrary element of $\mathbb{R}^3$. Then

$$\begin{aligned}
\lambda(\mathbf{u}) &= \lambda(u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}) && \text{(obviously)} \\
&= u_1\lambda(\hat{\mathbf{i}}) + u_2\lambda(\hat{\mathbf{j}}) + u_3\lambda(\hat{\mathbf{k}}) && \text{(since } \lambda \text{ is linear)} \\
&= u_1 w_1 + u_2 w_2 + u_3 w_3 && \text{(by definition of } \mathbf{w}\text{)} \\
&= \mathbf{u} \cdot \mathbf{w} \\
&= dot_{\mathbf{w}}(\mathbf{u}) && \text{(by definition of } dot_{\mathbf{w}}\text{)}
\end{aligned}$$

Since λ and $dot_{\mathbf{w}}$ agree on all inputs, they're equal. ◆

Dual Spaces

Extra homework

If you feel like it...
Finish the proof of the following theorem.

Theorem

Suppose $\lambda \in (\mathbb{R}^3)^*$. Then there exists *a unique vector* $\mathbf{w} \in \mathbb{R}^3$ such that $\lambda = dot_{\mathbf{w}}$.