Ma341 Differential Geometry: Fall 2025 Homework

Chapter 1: Calculus on Euclidean Space
1.1 Euclidean Space: 4.
1.2 Tangent Vectors: 1 (Use a CAS for part b), 2-5.
1.3 Directional Derivatives: 1a, 2a, 3df, 4, 5.
1.4 Curves in $\mathbb{R}^3$: 1 (Sketch with a CAS), 2, 4, 6, 7, 8a, 9.
1.5 1-Forms: 1b, 2, 3, 4c, 5b, 6a, 7, 9.
1.6 Differential Forms: 1, 2, 3, 4cd, 5, 6, 7, 8, 9.
1.7 Mappings: 2, 3, 4, 6a, 7, 8, 9b, 10.

Chapter 2: Frame Fields
2.1 Dot Product: 3, 6, 8, 10, 11a, 12.
2.2 Curves: 2, 6 (Sketch with a CAS), 9, A: Let $\alpha : (0,1) \to \mathbb{R}^2$ be the curve given by $\alpha(t) = (t,0)$, let Y be the vector field on α given by $Y(t) = (-1,1)_{\alpha(t)}$, and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by $F(u,v) = (u, u^2 + v)$. Compute and plot $(F_*(Y))'$. Does $(F_*(Y))' = F_*(Y')$?
2.3 The Frenet Formulas: 1, 2, 3.
2.4 Arbitrary-Speed Curves: 1 (Sketch with a CAS), 11, 17bd, 18, A: Find a parameterization for a cylindrical helix that passes through the origin at time $t = 0$, stays on the parabolic cylinder $M = \{(p_1, p_2, p_3) : p_2 = p_1^2\}$, and has a unit tangent T that maintains a 45 degree angle with $(0,0,1)$. Use a CAS to plot (part of) the route of the curve.
2.5 Covariant Derivatives: 1b, 2bf, 3, 4, 5.
2.6 Frame Fields: 1, 2, 3.
2.7 Connection Forms: 1, 2, 3, 5, 6, 7.
2.8 The Structural Equations: 1, 2, 3.

Chapter 3: Euclidean Geometry
3.1 Isometries of $\mathbb{R}^3$: 1, 2, 3, 4, 6, 9.
3.2 The Tangent Map of an Isometry: 3, 5.
3.3 Orientation: 1, 3, 5.
3.4 Euclidean Geometry: 4.
3.5 Congruence of Curves: 3, 4, 10a, A: Plot the space curve which satisfies $\beta(0) = (0,0,0)$, $\frac{d\beta}{ds}(0) = (1,0,0)$, $\kappa(s) = 2 + s^2 \cos^2 s$, and $\tau(s) = \sin s$, for $0 \le s \le 4\pi$.

Chapter 4: Calculus on a Surface
4.1 Surfaces in $\mathbb{R}^3$: 1, 4, 5, 8, 9, 10, 12, A: Let M be the unit sphere centered at the origin. Define $\mathbf{x} : \mathbb{R}^2 \to M \setminus \{(0,0,1)\}$ by $\mathbf{x}(u,v) =$(the unique point of $M \setminus \{(0,0,1)\}$ on the line through $(u,v,0)$ and $(0,0,1)$). Find explicit algebraic formulas for the coordinate functions of $\mathbf{x}$ and the inverse function $\mathbf{x}^{-1}$. Show that $\mathbf{x}$ is a proper patch on M., B: Find explicit algebraic formulas for the patches in the Monge patch atlas for the torus $M = \{(\sqrt{p_1^2 + p_2^2} - 2)^2 + p_3^2 = 1\}$. Make sure to specify the domain of each patch. Finally, sketch the domain of each patch.
4.2 Patch Computations: 1, 2, 3, 4, 5, 8, 9b.
4.3 Differentiable Functions and Tangent Vectors: 1a, 2, 4, 5, 6, 7, 10c (See prob. 9 for the definition of $\overline{T}_p(M)$), 11c, A: Validate the assumption in Definition 3.10.
4.4 Differential Forms on a Surface: 1, 2, 3, 4, 6, 7, A: Let $M = \{(p_1, p_2, p_3) : p_1^2 + p_2^2 = 1\}$, and let $\phi : TM \to \mathbb{R}$ be the 1-form on M given by $\phi(\mathbf{v}_p) = \mathbf{v}_p \bullet (-p_2 U_1(\mathbf{p}) + p_1 U_2(\mathbf{p}))$. Show that ϕ is closed, but not exact.

4.5 Mappings of Surfaces: 2, 3, 4, 5, 9a, A: In class we proved if $F : M \to N$ is a mapping of surfaces, and ξ is a 0-form on N, then $F^*(d\xi) = d(F^*\xi)$. Redo the proof using $\mathbf{x}_u$ and $\mathbf{x}_v$ (for an arbitrary patch $\mathbf{x}$ in M) in place of the arbitrary $\mathbf{v}$ in TM.
4.6 Integration of Forms: 1, 2, 3, 4, 7, 10.
4.7 Topological Properties of Surfaces: 1, 3, 4, 5, 8.
4.8 Manifolds: 1, 2, 3, 13.
4.8 Manifolds (optional problems): 6, 8(Fix the problem statement first.), 9, 10, 12, 14, 15, 16, For each of the following sets M, endow M with a differentiable structure by defining a suitable atlas. Indicate the dimension of the resulting manifold. A: Let M be the set of *all* lines in $\mathbb{R}^3$. B: Let M be the set of all lines in $\mathbb{R}^3$ that are tangent to the unit sphere. C: Let M be the set of all rotations about the origin in $\mathbb{R}^3$. (Does it make a significant difference whether or not we consider the null rotation [*i.e.* the identity map] to be an element of M?) D: Let $M = \{(x, y, z, a) \in \mathbb{R}^4 : x^2+y^2+z^2+a^2 = 1\}$.

Chapter 5: Shape Operators
5.1 The Shape Operator of $M \subset \mathbb{R}^3$: 1, 2, 3, 4, 6, 7, 9.
5.2 Normal Curvature: 1, 2, 3, 4, A: Find the principal curvatures and principal vectors at every point of a torus.
5.3 Gaussian Curvature: 1, 2, 3, 4, 7, 8.
5.4 Computational Techniques: 1, 2, 3, 5, 8, 16, 17, 18.
5.5 The Implicit Case: 4.
5.6 Special Curves in a Surface: 1, 2, 3a, 6, 7, 15.
5.7 Surfaces of Revolution: 1, 4, 7.

Chapter 6: Geometry of Surfaces in $\mathbb{R}^3$
6.1 The Fundamental Equations: 1, 2.
6.2 Form Computations: 1, 2.
6.3 Some Global Theorems: 1, 2, 5.
6.4 Isometries and Local Isometries: 1, 3, 8a,b, 10a,b, 13.
6.5 Intrinsic Geometry of Surfaces in $\mathbb{R}^3$: 1, 2, 4.
6.6 Orthogonal Coordinates: 1, 2, 3.
6.7 Integration and Orientation: 1, 3, 5, 6.
6.8 Total Curvature: 1, 2 (Use a CAS), 7, 10.
6.9 Congruence of Surfaces: 1, 5.

Chapter 7: Riemannian Geometry
7.6 The Gauss-Bonnet Theorem: 1c, 7, 8.