Simple surface example:
$M = \{(p_1, p_2, p_3) : p_1^2 + p_2^2 = 1\} \subset \mathbb{R}^3$.
Let $D = \{(u, v) : 0 < u^2 + v^2 < 2\}$ and define $\mathbf{x} : D \to \mathbb{R}^3$ by

$$\mathbf{x}(u, v) = \left(\frac{u}{\sqrt{u^2 + v^2}}, \frac{v}{\sqrt{u^2 + v^2}}, \tan\left(\frac{\pi}{2}(u^2 + v^2 - 1)\right) \right).$$

Claim: $\{\mathbf{x}\}$ is an atlas for M.

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(1) D is open.
(Obvious.)

(2) $\mathbf{x}$ is one-to-one.
Suppose $(u_1, v_1), (u_2, v_2) \in D$ and $\mathbf{x}(u_1, v_1) = \mathbf{x}(u_2, v_2)$.
Then (*) $(\frac{u_1}{\sqrt{u_1^2 + v_1^2}}, \frac{v_1}{\sqrt{u_1^2 + v_1^2}}) = (\frac{u_2}{\sqrt{u_2^2 + v_2^2}}, \frac{v_2}{\sqrt{u_2^2 + v_2^2}})$ and
(**) $\tan(\frac{\pi}{2}(u_1^2 + v_1^2 - 1)) = \tan(\frac{\pi}{2}(u_2^2 + v_2^2 - 1))$.
By (*) (u_1, v_1) and (u_2, v_2) normalize to the same point on the unit circle in the plane. This implies they lie on the same ray emanating from the origin. In other words, they have the same polar angle.
(**) implies $\frac{\pi}{2}(u_1^2 + v_1^2 - 1) = \frac{\pi}{2}(u_2^2 + v_2^2 - 1) + k\pi$ for some $k \in \mathbb{Z}$. But $(u_i, v_i) \in D$ implies $0 < u_i^2 + v_i^2 < 2$ which implies $-\frac{\pi}{2} < \frac{\pi}{2}(u_i^2 + v_i^2 - 1) < \frac{\pi}{2}$. So $k = 0$ and $u_1^2 + v_1^2 = u_2^2 + v_2^2$, *i.e.* (u_1, v_1) and (u_2, v_2) lie on a common circle centered at the origin, *i.e.* they have the same polar radius.
Thus $(u_1, v_1) = (u_2, v_2)$, and therefore $\mathbf{x}$ is one-to-one.

(3) $\mathbf{x}$ is regular.
Algebraic approach (#1)

$$J = \begin{pmatrix} \frac{\partial}{\partial u}\left(\frac{u}{\sqrt{u^2+v^2}}\right) & \frac{\partial}{\partial v}\left(\frac{u}{\sqrt{u^2+v^2}}\right) \\ \frac{\partial}{\partial u}\left(\frac{v}{\sqrt{u^2+v^2}}\right) & \frac{\partial}{\partial v}\left(\frac{v}{\sqrt{u^2+v^2}}\right) \\ \frac{\partial}{\partial u}(\tan(\frac{\pi}{2}(u^2+v^2-1))) & \frac{\partial}{\partial v}(\tan(\frac{\pi}{2}(u^2+v^2-1))) \end{pmatrix} = \begin{pmatrix} \frac{v^2}{(u^2+v^2)^{\frac{3}{2}}} & \frac{-uv}{(u^2+v^2)^{\frac{3}{2}}} \\ \frac{-uv}{(u^2+v^2)^{\frac{3}{2}}} & \frac{u^2}{(u^2+v^2)^{\frac{3}{2}}} \\ \pi u \sec^2(\frac{\pi}{2}(u^2+v^2-1)) & \pi v \sec^2(\frac{\pi}{2}(u^2+v^2-1)) \end{pmatrix}$$

$$det \begin{pmatrix} \text{row } 1 \\ \text{row } 3 \end{pmatrix} = \frac{\pi v \sec^2(\frac{\pi}{2}(u^2 + v^2 - 1))}{\sqrt{u^2 + v^2}}$$

which is only 0 when $v = 0$. So (at least) at points $(u, v) \in D$ with $v \neq 0$, $\mathbf{x}$ is regular.

$$det \begin{pmatrix} \text{row } 2 \\ \text{row } 3 \end{pmatrix} = \frac{-\pi u \sec^2(\frac{\pi}{2}(u^2 + v^2 - 1))}{\sqrt{u^2 + v^2}}$$

which is only 0 when $u = 0$. So at points with $u \neq 0$, $\mathbf{x}$ is regular.
But $(0, 0) \notin D$, so the observations above show that $\mathbf{x}$ is regular throughout D, as required.

Note that

$$det \begin{pmatrix} \text{row } 1 \\ \text{row } 2 \end{pmatrix} = 0$$

everywhere. This does not contradict the regularity claim. It just means that it's not possible to establish the claim at any points of D by considering merely the first two rows of J. In fact, this should be clear on geometric grounds. If the third component of $\mathbf{x}$ is discarded, the resulting map from $\mathbb{R}^2$ to $\mathbb{R}^2$ squashes all of D onto the unit circle, and is thus degenerate everywhere.

Not to belabor the point, but let $\tilde{\mathbf{x}} : D \to \mathbb{R}^2$ be given by $\tilde{\mathbf{x}}(u,v) = (\frac{u}{\sqrt{u^2+v^2}}, \frac{v}{\sqrt{u^2+v^2}})$, and suppose $\mathbf{p} \in D$. Now consider a curve α in D with $\alpha(0) = \mathbf{p}$. The image of α under $\tilde{\mathbf{x}}$ is, necessarily, (some parameterization of) an arc of the unit circle passing through $\tilde{\mathbf{x}}(\mathbf{p})$ at time 0, and thus $\tilde{\mathbf{x}}(\alpha)$ has an initial velocity tangent to the unit circle at $\tilde{\mathbf{x}}(\mathbf{p})$, ***regardless of the direction of*** $\alpha'(0)$. It follows that $\tilde{\mathbf{x}}_*$ maps all of $T_p\mathbb{R}^2$ into the one dimensional subspace of $T_{\tilde{\mathbf{x}}(p)}\mathbb{R}^2$ consisting of vectors at $\tilde{\mathbf{x}}(\mathbf{p})$ that are tangent to the unit circle at $\tilde{\mathbf{x}}(\mathbf{p})$.

(3) $\mathbf{x}$ is regular.
Algebraic approach (#2)

$$J = \begin{pmatrix} \frac{\partial}{\partial u}\left(\frac{u}{\sqrt{u^2+v^2}}\right) & \frac{\partial}{\partial v}\left(\frac{u}{\sqrt{u^2+v^2}}\right) \\ \frac{\partial}{\partial u}\left(\frac{v}{\sqrt{u^2+v^2}}\right) & \frac{\partial}{\partial v}\left(\frac{v}{\sqrt{u^2+v^2}}\right) \\ \frac{\partial}{\partial u}(\tan(\frac{\pi}{2}(u^2+v^2-1))) & \frac{\partial}{\partial v}(\tan(\frac{\pi}{2}(u^2+v^2-1))) \end{pmatrix} = \begin{pmatrix} \frac{v^2}{(u^2+v^2)^{\frac{3}{2}}} & \frac{-uv}{(u^2+v^2)^{\frac{3}{2}}} \\ \frac{-uv}{(u^2+v^2)^{\frac{3}{2}}} & \frac{u^2}{(u^2+v^2)^{\frac{3}{2}}} \\ \pi u \sec^2(\frac{\pi}{2}(u^2+v^2-1)) & \pi v \sec^2(\frac{\pi}{2}(u^2+v^2-1)) \end{pmatrix}$$

Consider first points $(u,v) \in D$ with $u \neq 0$.
Then, from the final entries in each column, we see that the only chance for the columns to be parallel is if (col 2) $= \frac{v}{u}$(col 1). But, then, considering the 2nd entries, this requires $-v^2 = u^2$, which doesn't hold anywhere in D. (Recall $(0,0) \notin D$.) So when $u \neq 0$, $rank(J) = 2$.
Now consider points of the form $(0,v) \in D$.
Then

$$J = \begin{pmatrix} \frac{1}{|v|} & 0 \\ 0 & 0 \\ 0 & \pi v \sec^2(\frac{\pi}{2}(v^2-1)) \end{pmatrix}$$

which clearly has rank 2.
So $\mathbf{x}$ is regular throughout D.

(3) $\mathbf{x}$ is regular.
Geometric approach
Suppose $\mathbf{p} = (u_0, v_0) \in D$.
Let $r_0 = \sqrt{u_0^2 + v_0^2}$, and let θ_0 be a polar angle for (u_0, v_0).
Define the curves α and β in D by $\alpha(t) = (r_0\cos(t-\theta_0), r_0\sin(t-\theta_0))$ and $\beta(t) = ((t+1)u_0, (t+1)v_0)$. (Note that $\alpha(0) = \beta(0) = \mathbf{p}$.)
The images of α and β under $\mathbf{x}$ are given by $\mathbf{x}(\alpha(t)) = (\cos(t-\theta_0), \sin(t-\theta_0), \tan(\frac{\pi}{2}(r_0^2-1)))$ and $\mathbf{x}(\beta(t)) = (\frac{u_0}{r_0}, \frac{v_0}{r_0}, \tan(\frac{\pi}{2}((t+1)^2r_0^2-1)))$. Their initial velocities are $(\mathbf{x}(\alpha))'(0) = (\sin\theta_0, \cos\theta_0, 0)$ and $(\mathbf{x}(\beta))'(0) = (0, 0, \frac{\pi}{2}r_0^2\sec^2(\frac{\pi}{2}(r_0^2-1)))$. These vectors are clearly linearly independent. It follows that the image of $\mathbf{x}_{*\mathbf{p}}$ is 2 dimensional (for all $\mathbf{p} \in D$). So $\mathbf{x}$ is regular.

(Notice that we've already established that $\mathbf{x}$ is a patch.)

(4) $\mathbf{x}(D) \subset M$

$\left(\frac{u}{\sqrt{u^2+v^2}}\right)^2 + \left(\frac{v}{\sqrt{u^2+v^2}}\right)^2$ clearly equals 1 for all $(u,v) \in D$. So $\mathbf{x}$ is a patch **in M**.

(5) $\mathbf{x}$ maps D ***onto*** M
Given $(p_1, p_2, p_3) \in M$, $(p_1\sqrt{1+\frac{2}{\pi}\arctan p_3}, p_2\sqrt{1+\frac{2}{\pi}\arctan p_3})$ is a point in D which gets mapped by $\mathbf{x}$ to (p_1, p_2, p_3).

(6) $\mathbf{x}$ is a proper patch
This follows since $\mathbf{x}^{-1} : M \to D$ given by $\mathbf{x}^{-1}((p_1, p_2, p_3)) = (p_1\sqrt{1 + \frac{2}{\pi}\arctan p_3}, p_2\sqrt{1 + \frac{2}{\pi}\arctan p_3})$ is continuous.

(7) M is a surface
Since the proper patch $\mathbf{x}$ covers all of M, it clearly covers a neighborhood of $\mathbf{p}$ in M for *every* $\mathbf{p} \in M$.

Furthermore, all the above shows that $\{\mathbf{x}\}$ is an atlas for M.