Lecture notes based on *Elementary Differential Geometry* by Barrett O'Neill.

3.1 Isometries of $\mathbb{R}^3$

1.1 Definition

An *isometry* of $\mathbb{R}^3$ is a mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$d(F(\mathbf{p}), F(\mathbf{q})) = d(\mathbf{p}, \mathbf{q})$$

for all points $\mathbf{p}$, $\mathbf{q}$ in $\mathbb{R}^3$. (In other words, a map that preserves distances.)

Examples

- Translations
- Rotations
- Reflections

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for all points $\mathbf{p}$, $\mathbf{q}$ in $\mathbb{R}^3$.

1.3 Lemma

If F and G are isometries of $\mathbb{R}^3$, then the composite mapping GF is also an isometry of $\mathbb{R}^3$.

Proof

Obvious and easy.

3.1 Isometries of $\mathbb{R}^3$

1.4 Lemma

(1) If S and T are translations, then $ST = TS$ is also a translation.
(2) If T is translation by $\mathbf{a}$, then T has an inverse T^{-1} which is translation by $-\mathbf{a}$.
(3) Given any two points $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$, there exists a unique translation T such that $T(\mathbf{p}) = \mathbf{q}$.
(In particular, if a translation has a fixed point, then it must be the identity, *i.e.* translation by $\mathbf{0}$.)

Proof

Obvious and easy.

3.1 Isometries of $\mathbb{R}^3$

Definition

A linear transformation $C : \mathbb{R}^3 \to \mathbb{R}^3$ is *orthogonal* if

$$C(\mathbf{p}) \cdot C(\mathbf{q}) = \mathbf{p} \cdot \mathbf{q} \text{ for all } \mathbf{p}, \mathbf{q}.$$

(In other words, if it preserves dot products.)

1.5 Lemma

If $C : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation, then C is an isometry of $\mathbb{R}^3$.

Proof

$$d(C(\mathbf{p}), C(\mathbf{q})) = ||C(\mathbf{p}) - C(\mathbf{q})|| = ||C(\mathbf{p} - \mathbf{q})|| = \sqrt{C(\mathbf{p} - \mathbf{q}) \cdot C(\mathbf{p} - \mathbf{q})}$$
$$= \sqrt{(\mathbf{p} - \mathbf{q}) \cdot (\mathbf{p} - \mathbf{q})} = ||\mathbf{p} - \mathbf{q}|| = d(\mathbf{p}, \mathbf{q}) \quad \blacklozenge$$

3.1 Isometries of $\mathbb{R}^3$

1.6 Lemma

If F is an isometry of $\mathbb{R}^3$ such that $F(\mathbf{0}) = \mathbf{0}$, then F is an orthogonal transformation.

Proof

Suppose F is an isometry that leaves the origin fixed.
Then $||F(\mathbf{p})|| = d(\mathbf{0}, F(\mathbf{p})) = d(F(\mathbf{0}), F(\mathbf{p})) = d(\mathbf{0}, \mathbf{p}) = ||\mathbf{p}||$.
So F preserves norms (not just norms of differences).
Now suppose $\mathbf{p}$ and $\mathbf{q}$ are an arbitrary pair of points.
Then $||F(\mathbf{p}) - F(\mathbf{q})|| = ||\mathbf{p} - \mathbf{q}||$.
So $(F(\mathbf{p}) - F(\mathbf{q})) \cdot (F(\mathbf{p}) - F(\mathbf{q})) = (\mathbf{p} - \mathbf{q}) \cdot (\mathbf{p} - \mathbf{q})$.
Thus $||F(\mathbf{p})||^2 - 2F(\mathbf{p}) \cdot F(\mathbf{q}) + ||F(\mathbf{q})||^2 = ||\mathbf{p}||^2 - 2\mathbf{p} \cdot \mathbf{q} + ||\mathbf{q}||^2$.
Since F preserves norms, this simplifies to $F(\mathbf{p}) \cdot F(\mathbf{q}) = \mathbf{p} \cdot \mathbf{q}$, showing that F does indeed preserve dot products.

1.6 Lemma

If F is an isometry of $\mathbb{R}^3$ such that $F(\mathbf{0}) = \mathbf{0}$, then F is an orthogonal transformation.

Proof (cont.)

Let $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ be the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, respectively.
Then $\mathbf{p} = \sum p_i \mathbf{u}_i$ and $\mathbf{u}_i \bullet \mathbf{u}_j = \delta_{ij}$.
Since F preserves dot products, it follows that $F(\mathbf{u}_i) \bullet F(\mathbf{u}_j) = \delta_{ij}$.
(In other words, $F(\mathbf{u}_1)$, $F(\mathbf{u}_2)$, $F(\mathbf{u}_3)$ is orthonormal.)
So $F(\mathbf{p}) = \sum (F(\mathbf{p}) \bullet F(\mathbf{u}_i)) F(\mathbf{u}_i) = \sum (\mathbf{p} \bullet \mathbf{u}_i) F(\mathbf{u}_i) = \sum p_i F(\mathbf{u}_i)$.
Linearity of F now follows easily. ◆

1.7 Theorem

If F is an isometry of $\mathbb{R}^3$, then there exists a unique translation T and a unique orthogonal transformation C such that $F = TC$.

Proof (Existence)

Let T be translation by $F(\mathbf{0})$.
Then $T^{-1}F$ is an isometry, and

$$(T^{-1}F)(\mathbf{0}) = T^{-1}(F(\mathbf{0})) = F(\mathbf{0}) - F(\mathbf{0}) = \mathbf{0}.$$

So, by Lemma 1.6, $C = T^{-1}F$ is an orthogonal transformation.
Clearly $F = TC$. ◆

Proof (Uniqueness)

See text.

3.1 Isometries of $\mathbb{R}^3$

1.7 Theorem (Alternative form)

If F is an isometry of $\mathbb{R}^3$, then there exists a unique set of scalars a_i, $(1 \leq i \leq 3)$, c_{ij}, $(1 \leq i, j \leq 3)$ such that $\sum_k c_{ki} c_{kj} = \delta_{ij}$ and $\mathbf{q} = F(\mathbf{p})$ may be computed by

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

3.2 The Tangent Map of an Isometry

2.1 Theorem

If F is an isometry of $\mathbb{R}^3$ with orthogonal part C, then

$$F_*(\mathbf{v}_p) = C(\mathbf{v})_{F(p)}$$

for all $\mathbf{v}_p \in T\mathbb{R}^3$.

2.2 Corollary

If F is an isometry, then

$$F_*(\mathbf{v}_p) \cdot F_*(\mathbf{w}_p) = \mathbf{v}_p \cdot \mathbf{w}_p$$

Corollary of corollary

If F is an isometry and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a frame at point $\mathbf{p}$, then $(F(\mathbf{e}_1), F(\mathbf{e}_2), F(\mathbf{e}_3))$ is a frame at $F(\mathbf{p})$.

2.3 Theorem

Given any two frames, say $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ at $\mathbf{p}$ and $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ and $\mathbf{q}$, there exists a unique isometry F such that $F_*(\mathbf{e}_i) = \mathbf{f}_i$ for $1 \leq i \leq 3$.

Theorem

Given a frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ with attitude matrix A,

$$\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = det(A) = \pm 1.$$

Proof

Exercise 2.1.6 (and the basic lin alg result equating scalar triple products with determinants)

Definition

When $\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = det(A) = 1$ we say the frame is *positively oriented*, and when $\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = det(A) = -1$ we say the frame is *negatively oriented*.

Observations

- For any $\mathbf{p} \in \mathbb{R}^3$, $(U_1(\mathbf{p}), U_2(\mathbf{p}), U_3(\mathbf{p}))$ is positively oriented.
- Frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is positively oriented iff $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$.
- Frenet frames are always positively oriented.

Definition

The *sign* of an isometry F (written $sgn(F)$) is the determinant of the orthogonal part of F.

3.2 Lemma

Given a frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and an isometry F,

$$F_*(\mathbf{e}_1) \bullet F_*(\mathbf{e}_2) \times F_*(\mathbf{e}_3) = (sgn\, F)\mathbf{e}_1 \bullet \mathbf{e}_2 \times \mathbf{e}_3.$$

And so we make the following definitions.

3.3 Definition

An isometry F is called *orientation-preserving* if $sgn\, F = +1$, and *orientation-reversing* if $sgn\, F = -1$.

Translations and rotations are always orientation-preserving. Reflections are always orientation-reversing.

3.3 Orientation

3.5 Lemma

Suppose $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a frame, $\mathbf{v} = \sum v_i \mathbf{e}_i$ and $\mathbf{w} = \sum w_i \mathbf{e}_i$. Then

$$\mathbf{v} \times \mathbf{w} = \mathbf{e}_1 \bullet \mathbf{e}_2 \times \mathbf{e}_3 \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

3.6 Theorem

If F is an isometry, and $\mathbf{v}$ and $\mathbf{w}$ are tangent vectors at the same point, then

$$F_*(\mathbf{v} \times \mathbf{w}) = (sgn\, F) F_*(\mathbf{v}) \times F_*(\mathbf{w}).$$

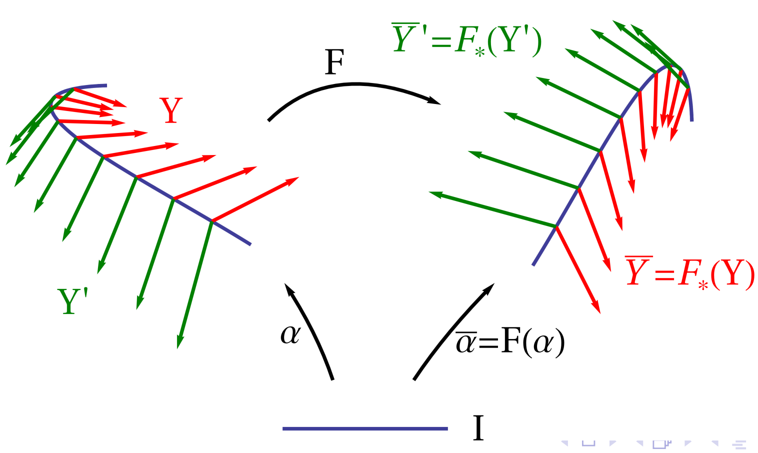
Q: When is a property or construct associated with a region, curve, vector field, or other mathematical object in $\mathbb{R}^3$ "geometric"?

A: Euclidean geometry is the collection of properties and constructs that are preserved by all isometries of Euclidean space, but not by all diffeomorphisms.

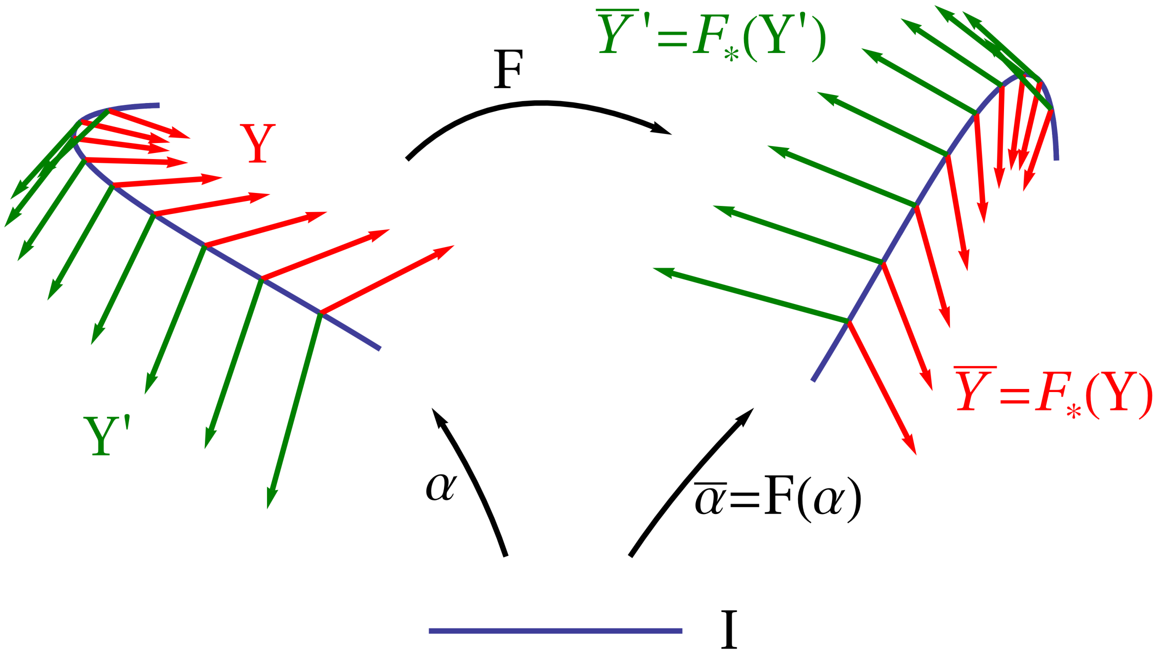
4.1 Corollary (of Thm 2.1)

Suppose Y is a vector field on curve α and F is an isometry. Then $F_*(Y)$ is a vector field on $F(\alpha)$, and

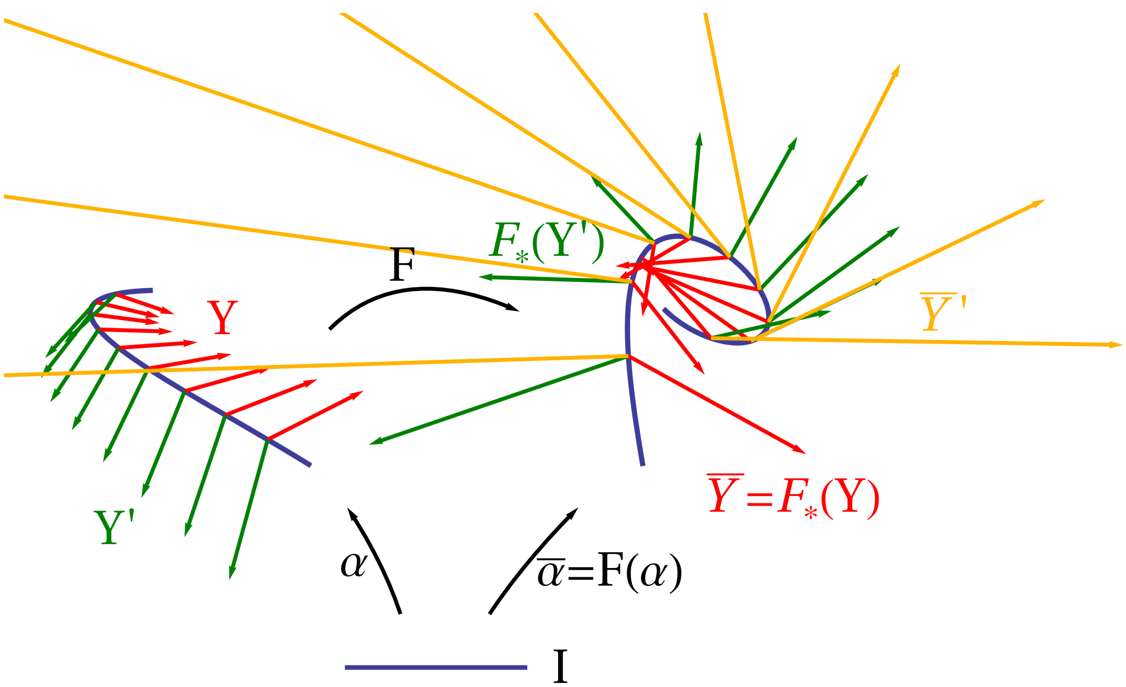
$$(F_*(Y))' = F_*(Y').$$



3.4 Euclidean Geometry



3.4 Euclidean Geometry



3.4 Euclidean Geometry

Proof

See text.

Corollary of corollary

Isometries preserve acceleration.

4.2 Theorem

Suppose β is a unit-speed curve (with positive curvature), and F is an isometry. Let $\bar{\beta} = F(\beta)$. Then

$$\bar{\kappa} = \kappa, \qquad \bar{\tau} = (sgn\, F)\tau,$$

$$\bar{T} = F_*(T), \qquad \bar{N} = F_*(N), \qquad \bar{B} = (sgn\, F)F_*(B).$$

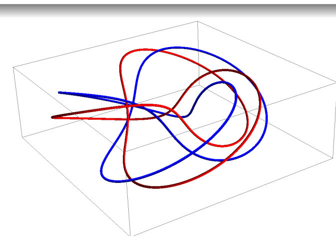
3.5 Congruence of Curves

5.1 Definition

Curves α and β are *congruent* if there exists an isometry F such that $\beta = F(\alpha)$.

Example

The trefoil knots $\alpha(t) = ((2 + \cos 3t)\cos 2t, (2 + \cos 3t)\sin 2t, \sin 3t)$ and $\beta(t) = (-(2 + \cos 3t)\sin 2t, (2 + \cos 3t)\cos 2t, \sin 3t)$ are congruent, since $\beta = F(\alpha)$ for F given by $F(p_1, p_2, p_3) = (-p_2, p_1, p3)$ (90° rotation around the z-axis).



5.3 Theorem

If $\alpha, \beta : I \to \mathbb{R}^3$ are unit-speed curves such that $\kappa_\alpha = \kappa_\beta$ and $\tau_\alpha = \pm\tau_\beta$, then α and β are congruent.

Sketch of proof

Choose some $s_0 \in I$.
Let F be the (uniquely determined) isometry that takes the Frenet frame $(T_\alpha(s_0), N_\alpha(s_0), B_\alpha(s_0))$ at $\alpha(s_0)$ to $(T_\beta(s_0), N_\beta(s_0), \pm B_\beta(s_0))$ at $\beta(s_0)$ (where the plus sign is used if $\tau_\alpha = \tau_\beta$ and the minus sign if $\tau_\alpha = -\tau_\beta$).
Define $\bar{\alpha} = F(\alpha)$. (The plan is to show that $\bar{\alpha} = \beta$.)
Theorem 4.2 (and the construction of F) immediately yield

$$\bar{\alpha}(s_0) = \beta(s_0),\ T_{\bar{\alpha}}(s_0) = T_\beta(s_0),\ N_{\bar{\alpha}}(s_0) = N_\beta(s_0),\ B_{\bar{\alpha}}(s_0) = B_\beta(s_0)$$

$$\kappa_{\bar{\alpha}} = \kappa_\beta,\ \tau_{\bar{\alpha}} = \tau_\beta.$$

$$\bar{\alpha}(s_0) = \beta(s_0),\ T_{\bar{\alpha}}(s_0) = T_\beta(s_0),\ N_{\bar{\alpha}}(s_0) = N_\beta(s_0),\ B_{\bar{\alpha}}(s_0) = B_\beta(s_0)$$

$$\kappa_{\bar{\alpha}} = \kappa_\beta,\ \tau_{\bar{\alpha}} = \tau_\beta.$$

Now the clever part.
Consider the (real-valued) function $f = T_{\bar{\alpha}} \cdot T_\beta + N_{\bar{\alpha}} \cdot N_\beta + B_{\bar{\alpha}} \cdot B_\beta$ on I.
Claim: f is constant on I.
Compute

$$\begin{aligned} f' &= T'_{\bar{\alpha}} \cdot T_\beta + T_{\bar{\alpha}} \cdot T'_\beta + N'_{\bar{\alpha}} \cdot N_\beta + N_{\bar{\alpha}} \cdot N'_\beta + B'_{\bar{\alpha}} \cdot B_\beta + B_{\bar{\alpha}} \cdot B'_\beta \\ &= (\kappa_{\bar{\alpha}} N_{\bar{\alpha}}) \cdot T_\beta + T_{\bar{\alpha}} \cdot (\kappa_\beta N_\beta) + (-\kappa_{\bar{\alpha}} T_{\bar{\alpha}} + \tau_{\bar{\alpha}} B_{\bar{\alpha}}) \cdot N_\beta \\ &\quad + N_{\bar{\alpha}} \cdot (-\kappa_\beta T_\beta + \tau_\beta B_\beta) + (-\tau_{\bar{\alpha}} N_{\bar{\alpha}}) \cdot B_\beta + B_{\bar{\alpha}} \cdot (-\tau_\beta N_\beta) \\ &= 0. \end{aligned}$$

Q: What constant value does $f = T_{\bar{\alpha}} \cdot T_\beta + N_{\bar{\alpha}} \cdot N_\beta + B_{\bar{\alpha}} \cdot B_\beta$ hold over I?

A: $f(s) = f(s_0) = T_{\bar{\alpha}}(s_0) \cdot T_\beta(s_0) + N_{\bar{\alpha}}(s_0) \cdot N_\beta(s_0) + B_{\bar{\alpha}}(s_0) \cdot B_\beta(s_0)$
$= 1 + 1 + 1 = 3$ (for all $s \in I$).

But this implies $T_{\bar{\alpha}}(s) = T_\beta(s)$, $N_{\bar{\alpha}}(s) = N_\beta(s)$, $B_{\bar{\alpha}}(s) = B_\beta(s)$, for all $s \in I$.

Finally $\beta(s) = \beta(s_0) + \int_{s_0}^{s} \beta'(\sigma)d\sigma = \beta(s_0) + \int_{s_0}^{s} T_\beta(\sigma)d\sigma$

$$= \bar{\alpha}(s_0) + \int_{s_0}^{s} T_{\bar{\alpha}}(\sigma)d\sigma = \bar{\alpha}(s) \qquad \text{(for all } s \in I\text{)}. \qquad \blacklozenge$$

3.5 Congruence of Curves

5.5 Corollary

A unit speed curve is a helix iff both its curvature and torsion are nonzero constants.