Chapter 1

Lecture notes based on *Elementary Differential Geometry* by Barrett O'Neill.

1.1 Euclidean Space

1.2 Definition

Let x, y, and z be the real-valued functions on $\mathbb{R}^3$ such that for each point $\mathbf{p} = (p_1, p_2, p_3)$

$$x(\mathbf{p}) = p_1, \;\; y(\mathbf{p}) = p_2, \;\; z(\mathbf{p}) = p_3.$$

These functions are called the *natural coordinate functions* of $\mathbb{R}^3$.

(Note: Sometimes we write x_1, x_2, and x_3 instead of x, y, and z.)
Any point $\mathbf{p}$ can be reconstituted from its images under the natural coordinate functions.

$$\mathbf{p} = (p_1, p_2, p_3) = (x(\mathbf{p}), y(\mathbf{p}), z(\mathbf{p})) = (x_1(\mathbf{p}), x_2(\mathbf{p}), x_3(\mathbf{p}))$$

1.1 Euclidean Space

1.3 Definition

A real-valued function f on $\mathbb{R}^3$ is *of class* C^∞ provided all partial derivatives of f, of all orders, exist and are continuous.

In analysis, the term *differentiable* usually means existence of a first order derivative.
In this class, the default meaning of *differentiable* (or *smooth*) will be C^∞.
In fact, the default meaning of *function* will be *real-valued* C^∞ *function*.

1.2 Tangent Vectors

2.1 Definition

A *tangent vector* $\mathbf{v}_p$ to $\mathbb{R}^3$ consists of two points of $\mathbb{R}^3$: its *vector part* $\mathbf{v}$ and its *point of application* $\mathbf{p}$.

$$\mathbf{v}_p = \mathbf{w}_q \text{ iff } (\mathbf{p} = \mathbf{q} \text{ and } \mathbf{v} = \mathbf{w})$$

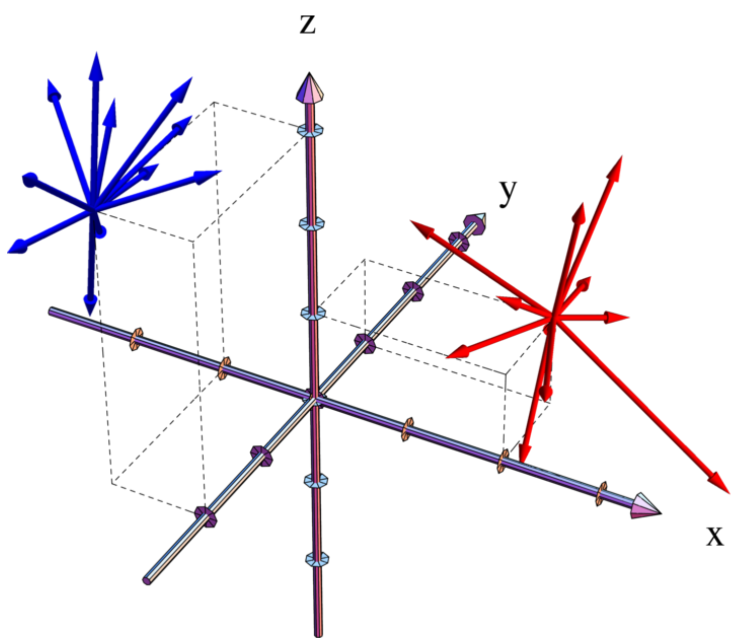
2.2 Definition

Let $\mathbf{p}$ be a point of $\mathbb{R}^3$. The set $T_p(\mathbb{R}^3)$ consisting of all tangent vectors with point of application $\mathbf{p}$ is called the *tangent space* of $\mathbb{R}^3$ at $\mathbf{p}$.

$T_p(\mathbb{R}^3)$ is a vector space with addition and scaling defined by

$$\mathbf{v}_p + \mathbf{w}_p = (\mathbf{v} + \mathbf{w})_p, \quad c(\mathbf{v}_p) = (c\mathbf{v})_p.$$

1.2 Tangent Vectors



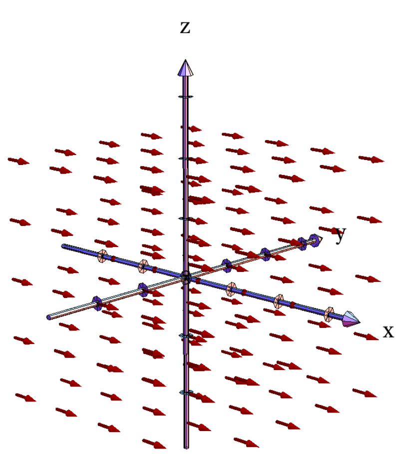
Tangent vectors at different points live in different vector spaces.

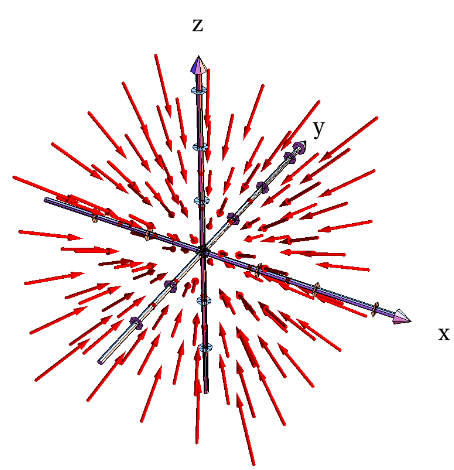
1.2 Tangent Vectors

2.3 Definition

A *vector field* on $\mathbb{R}^3$ is a function that assigns to each point $\mathbf{p}$ of $\mathbb{R}^3$ a tangent vector $V(\mathbf{p})$ to $\mathbb{R}^3$ at $\mathbf{p}$.

In other words, a vector field is a selection of one vector from each tangent space.





1.2 Tangent Vectors

Suppose V and W are vector fields on $\mathbb{R}^3$.
Then at each point $\mathbf{p} \in \mathbb{R}^3$, $V(\mathbf{p})$ and $W(\mathbf{p})$ are both elements of the same tangent space, namely $T_p\mathbb{R}^3$, and thus may be added to produce a new tangent vector, also in $T_p\mathbb{R}^3$.
So V and W can be added "pointwise" to produce a new vector field $V + W$ such that ...

$$(V + W)(\mathbf{p}) = V(\mathbf{p}) + W(\mathbf{p})$$

at each point $\mathbf{p}$.

This is an example of the *pointwise principle*.

1.2 Tangent Vectors

Pointwise principle

If an operation can be performed on the values of two functions at each point, then that operation can be extended to the functions themselves: simply apply it to their values at each point.

Vector field V can be scaled, pointwise, by scalar c, so that

$$(cV)(\mathbf{p}) = c(V(\mathbf{p})) \quad \text{for all } \mathbf{p}.$$

But we can also do something more interesting and flexible than uniform scaling. There's no need to scale by the same factor at each point. In other words, vector field V can be scaled by any real-valued function f on $\mathbb{R}^3$, so that ...

$$(f\,V)(\mathbf{p}) = f(\mathbf{p})V(\mathbf{p}) \quad \text{for all } \mathbf{p}.$$

1.2 Tangent Vectors

2.4 Definition: Natural frame field

Let U_1, U_2, and U_3 be the vector fields on $\mathbb{R}^3$ such that

$$U_1(\mathbf{p}) = (1, 0, 0)_p$$

$$U_2(\mathbf{p}) = (0, 1, 0)_p$$

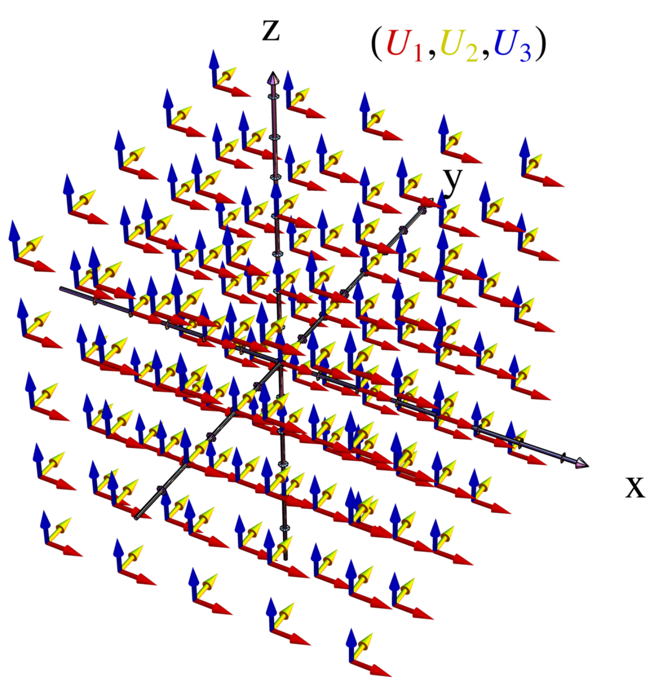
$$U_3(\mathbf{p}) = (0, 0, 1)_p$$

for each point $\mathbf{p}$ of $\mathbb{R}^3$.

The ordered triple (U_1, U_2, U_3) is called the *natural frame field* on $\mathbb{R}^3$.

1.2 Tangent Vectors

The natural frame field on $\mathbb{R}^3$



1.2 Tangent Vectors

2.5 Lemma

If V is a vector field on $\mathbb{R}^3$, there are three uniquely determined real-valued functions, v_1, v_2, v_3 on $\mathbb{R}^3$ such that

$$V = v_1 U_1 + v_2 U_2 + v_3 U_3.$$

The functions v_1, v_2, v_3 are called the *Euclidean coordinate functions* of V.

Proof

See text...or handout.

1.3 Directional Derivatives

3.1 Definition

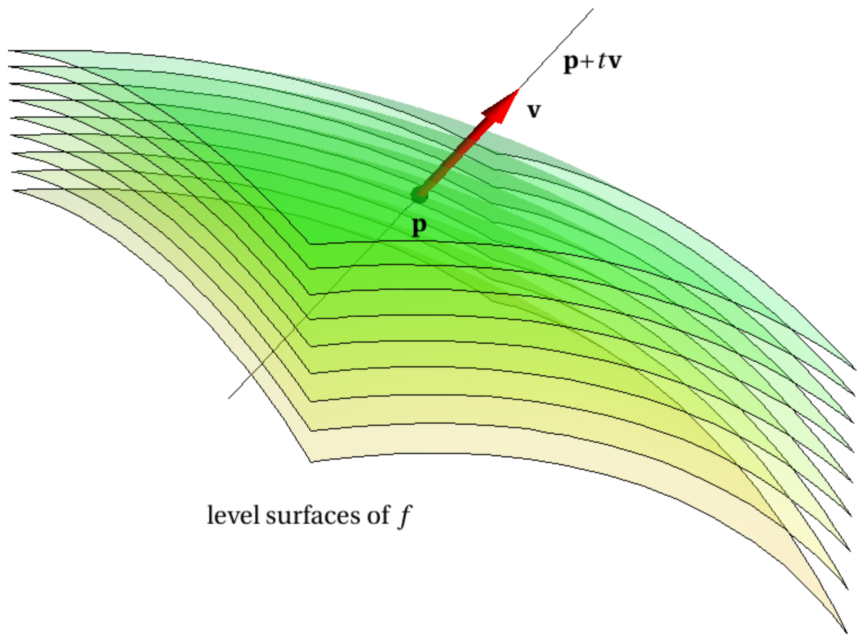
Let f be a differentiable real-valued function on $\mathbb{R}^3$, and let $\mathbf{v}_p$ be a tangent vector to $\mathbb{R}^3$. Then

$$\mathbf{v}_p[f] = \frac{d}{dt}(f(\mathbf{p} + t\mathbf{v}))\Big|_{t=0}$$

is called the *derivative of f with respect to $\mathbf{v}_p$*.

Notice that we do *not* require $\mathbf{v}_p$ to be a unit vector.

1.3 Directional Derivatives



1.3 Directional Derivatives

3.2 Lemma

If $\mathbf{v}_p = (v_1, v_2, v_3)_p$ is a tangent vector to $\mathbb{R}^3$, then

$$\mathbf{v}_p[f] = \sum v_i \frac{\partial f}{\partial x_i}(\mathbf{p}).$$

Proof

$$\begin{aligned} \mathbf{v}_p[f] &= \frac{d}{dt}(f(\mathbf{p}+t\mathbf{v}))\Big|_{t=0} && \text{(definition)} \\ &= \sum \frac{\partial f}{\partial x_i}(\mathbf{p}+t\mathbf{v})\frac{d}{dt}(p_i+tv_i)\Big|_{t=0} && \text{(chain rule)} \\ &= \sum \frac{\partial f}{\partial x_i}(\mathbf{p})v_i && \blacklozenge \end{aligned}$$

1.3 Directional Derivatives

3.3 Theorem

Let f and g be functions on $\mathbb{R}^3$, $\mathbf{v}_p$ and $\mathbf{w}_p$ tangent vectors, a and b scalars. Then

$$(a\,\mathbf{v}_p + b\,\mathbf{w}_p)[f] = a\,\mathbf{v}_p[f] + b\,\mathbf{w}_p[f], \tag{1}$$

$$\mathbf{v}_p[a\,f + b\,g] = a\,\mathbf{v}_p[f] + b\,\mathbf{v}_p[g], \text{ and} \tag{2}$$

$$\mathbf{v}_p[f\,g] = \mathbf{v}_p[f]\,g(\mathbf{p}) + f(\mathbf{p})\,\mathbf{v}_p[g]. \tag{3}$$

Proof

See text.

1.3 Directional Derivatives

Now suppose that V is a vector field on $\mathbb{R}^3$ and f is a real-valued function on $\mathbb{R}^3$.
At any point $\mathbf{p} \in \mathbb{R}^3$ we can use the tangent vector $V(\mathbf{p})$ to compute a directional derivative of f at $\mathbf{p}$.
This directional derivative, $V(\mathbf{p})[f]$, has a numerical value.
So $V[f]$ defines a function on $\mathbb{R}^3$ in a natural way as follows.
For each $\mathbf{p} \in \mathbb{R}^3$,

$$(V[f])(\mathbf{p}) = V(\mathbf{p})[f].$$

In other words, a vector field specifies a family of directional derivative operators – one for each point.
Operating on a function with a vector field produces a new function by differentiating the first function, everywhere, in the directions determined by the vector field.

1.3 Directional Derivatives

Question

Why not define $V[f]$ by

$$(V[f])(\mathbf{p}) = V(\mathbf{p})[f(\mathbf{p})] \text{ ?}$$

1.3 Directional Derivatives

Example

$$\begin{aligned}
(U_1[f])(\mathbf{p}) &= U_1(\mathbf{p})[f] && \text{(def. of vector field operating on fcn.)}\\
&= \tfrac{d}{dt}(f(\mathbf{p} + tU_1(\mathbf{p})))\Big|_{t=0} && \text{(def. of directional derivative)}\\
&= \tfrac{d}{dt}(f(p_1 + t, p_2, p_3))\Big|_{t=0} && \text{(def. of } U_1\text{)}\\
&= \tfrac{\partial f}{\partial x_1}(\mathbf{p}) && \text{(def. of partial derivative)}
\end{aligned}$$

Since $(U_1[f])(\mathbf{p}) = \dfrac{\partial f}{\partial x_1}(\mathbf{p})$ at all points $\mathbf{p}$, we conclude $U_1[f] = \dfrac{\partial f}{\partial x_1}$.

Similarly, $U_2[f] = \dfrac{\partial f}{\partial x_2}$ and $U_3[f] = \dfrac{\partial f}{\partial x_3}$.

1.3 Directional Derivatives

3.4 Corollary

If V and W are vector fields on $\mathbb{R}^3$ and f, g, h are real-valued functions, then

(1) $(f\,V + g\,W)[h] = f\,V[h] + g\,W[h]$,

(2) $V[a\,f + b\,g] = a\,V[f] + b\,V[g]$, for all scalars a and b, and

(3) $V[fg] = V[f]\,g + f\,V[g]$.

Proof

Follows almost immediately from Theorem 3.3 and the pointwise principle. See text.

1.3 Directional Derivatives

3.4 Corollary

If V and W are vector fields on $\mathbb{R}^3$ and f, g, h are real-valued functions, then

(1) $(f\,V + g\,W)[h] = f\,V[h] + g\,W[h]$,

(2) $V[a\,f + b\,g] = a\,V[f] + b\,V[g]$, for all scalars a and b, and

(3) $V[fg] = V[f]\,g + f\,V[g]$.

Example

Given $V = xy^2 U_2 - 2xU_3$ and $f = 3xz^2 - y^3 z$.

$$V[f] \overset{(2)}{=} 3V[xz^2] - V[y^3 z] \overset{(1)}{=} 3(xy^2 U_2[xz^2] - 2xU_3[xz^2]) - (xy^2 U_2[y^3 z] - 2xU_3[y^3 z])$$
$$= 3(xy^2(0) - 2x(2xz)) - (xy^2(3y^2 z) - 2x(y^3)) = -12x^2 z - 3xy^4 z + 2xy^3.$$

1.3 Directional Derivatives

3.5 Announcement

Henceforth, the point of application subscript will often by omitted from the notation for tangent vectors.

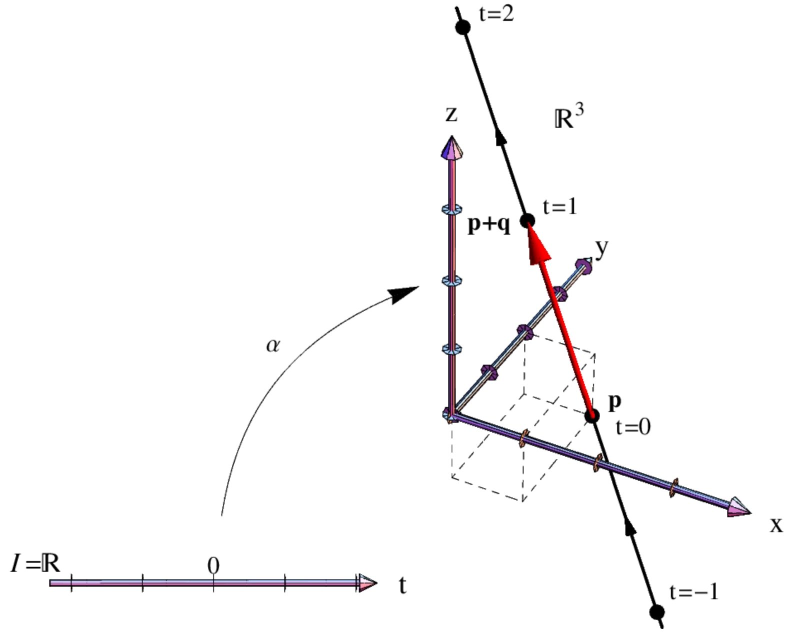
1.4 Curves in $\mathbb{R}^3$

4.1 Definition

A *curve* in $\mathbb{R}^3$ is a differentiable function $\alpha : I \to \mathbb{R}^3$ from an open interval I into $\mathbb{R}^3$.

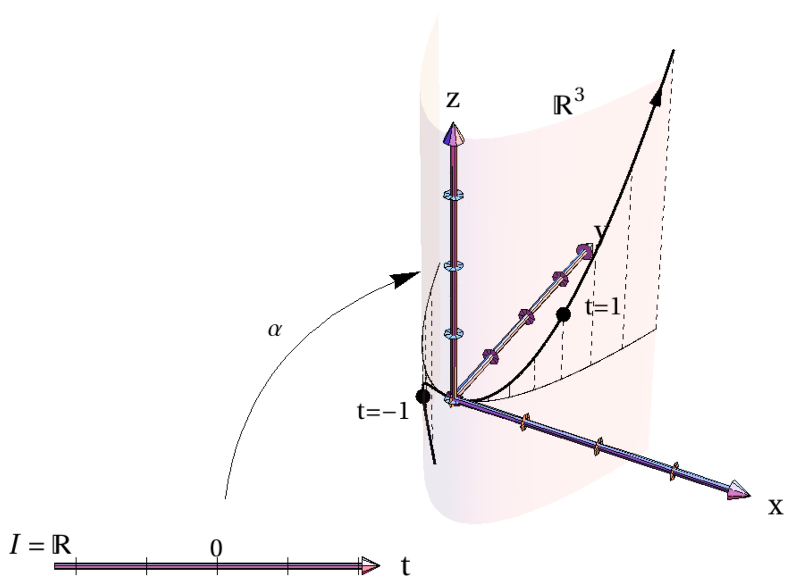
So by default "curve" means parameterized curve.
If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, then α_1, α_2, and α_3 are called the *Euclidean coordinate functions* of α.

1.4 Curves in $\mathbb{R}^3$



Line $\alpha : \mathbb{R} \to \mathbb{R}^3$ with $\alpha(t) = \mathbf{p} + t\mathbf{q}$, $(\mathbf{p}, \mathbf{q} \in \mathbb{R}^3, \mathbf{q} \neq \mathbf{0})$

1.4 Curves in $\mathbb{R}^3$



Twisted cubic $\alpha : \mathbb{R} \to \mathbb{R}^3$ with $\alpha(t) = (t, t^2, t^3)$

1.4 Curves in $\mathbb{R}^3$

4.3 Definition

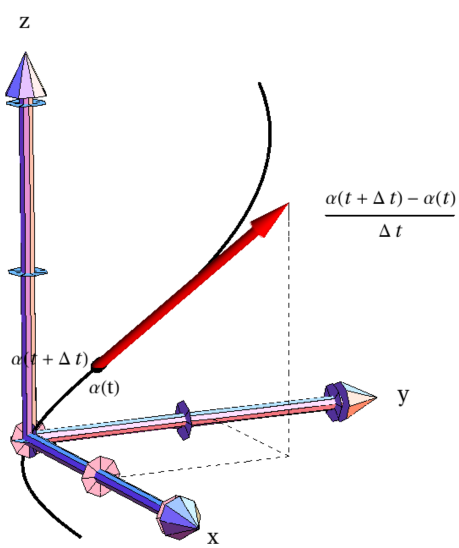
Let $\alpha : I \to \mathbb{R}^3$ be a curve in $\mathbb{R}^3$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For each $t \in I$, the *velocity vector of* α *at* t is the tangent vector

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}(t), \frac{d\alpha_2}{dt}(t), \frac{d\alpha_3}{dt}(t)\right)_{\alpha(t)}$$

$$= \sum \frac{d\alpha_i}{dt}(t) U_i(\alpha(t))$$

at the point $\alpha(t)$ in $\mathbb{R}^3$.

1.4 Curves in $\mathbb{R}^3$



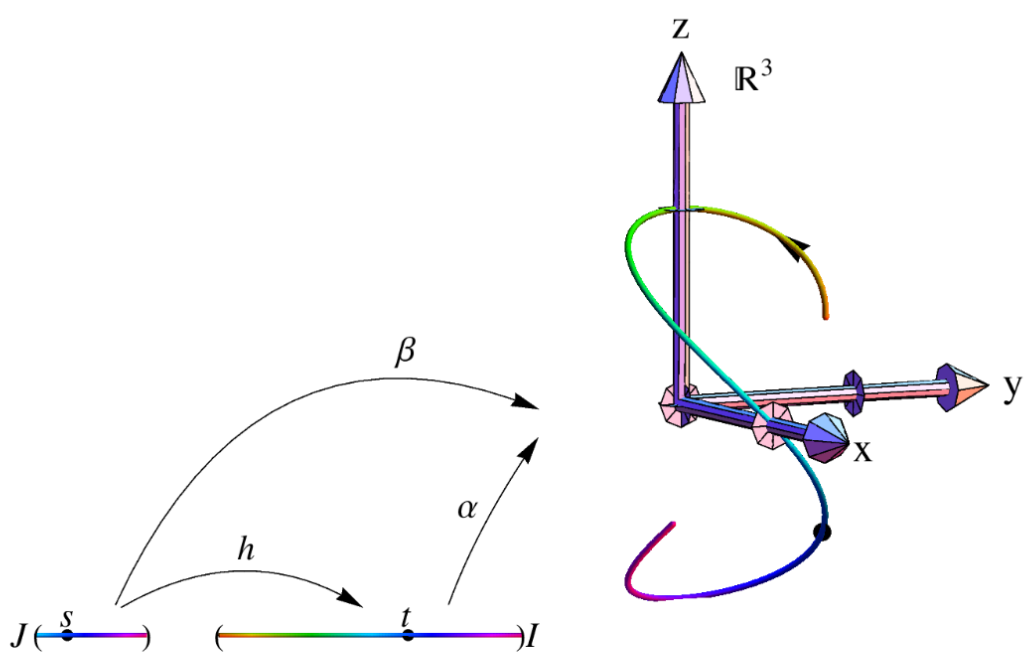
4.4 Definition

Let $\alpha : I \to \mathbb{R}^3$ be a curve. If $h : J \to I$ is a differentiable function on an open interval J, then the composite function

$$\beta = \alpha(h) : J \to \mathbb{R}^3$$

is a curve called a *reparameterization* of α by h.

For each $s \in J$, the new curve β is at the point $\beta(s) = \alpha(h(s))$ reached by α at $h(s)$ in I. So β travels the portion of the trace of α corresponding to $h(J) \subset I$.



4.4 Definition

Let $\alpha : I \to \mathbb{R}^3$ be a curve. If $h : J \to I$ is a differentiable function on an open interval J, then the composite function

$$\beta = \alpha(h) : J \to \mathbb{R}^3$$

is a curve called a *reparameterization* of α by h.

Example

$$\begin{aligned} \alpha(t) &= (\sin t, \cos t, 0), & -\pi < t < \pi \\ h(s) &= \arcsin s, & -1 < s < 1 \\ \beta(s) &= \alpha(h(s)) = (s, \sqrt{1-s^2}, 0), & -1 < s < 1 \end{aligned}$$

4.5 Lemma

If β is the reparameterization of α by h, then

$$\beta'(s) = \frac{dh}{ds}(s)\,\alpha'(h(s)).$$

Proof

Apply the chain rule.

1.4 Curves in $\mathbb{R}^3$

4.6 Lemma

Let α be a curve in $\mathbb{R}^3$ and let f be a differentiable function on $\mathbb{R}^3$. Then

$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t).$$

Proof

Since $\alpha' = \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \frac{d\alpha_3}{dt}\right)_\alpha$, Lemma 3.2 yields

$$\alpha'(t)[f] = \sum \frac{\partial f}{\partial x_i}(\alpha(t))\frac{d\alpha_i}{dt}(t).$$

But this is exactly the expression the chain rule produces for $\frac{d(f(\alpha))}{dt}(t)$.◆

1.4 Curves in $\mathbb{R}^3$

4.6 Lemma

Let α be a curve in $\mathbb{R}^3$ and let f be a differentiable function on $\mathbb{R}^3$. Then

$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t).$$

Question

How should we interpret Lemma 4.6?

1.4 Curves in $\mathbb{R}^3$

Definition

A curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ is *periodic* if there exists a number $p > 0$ such that $\alpha(t) = \alpha(t + p)$ for all t. The smallest such p (if it exists) is called the *period* of α.

Definition

A curve α is *regular* if α' never has a zero vector part.

1.4 Curves in $\mathbb{R}^3$

If f is a differentiable real-valued function on $\mathbb{R}^2$, let

$$C : f = a$$

be the set of all points $\mathbf{p}$ in $\mathbb{R}^2$ such that $f(\mathbf{p}) = a$.

Claim

If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are never simultaneously zero at any point of C, then C consists of one or more separate "components," each of which corresponds to the "route" taken by (many) regular curves.

Definition

We call these components *Curves* (with a capital 'C').

Definition

A regular curve α whose route matches a Curve C is called a *parameterization* of C.