Advanced Calculus Review / Primer

Calc I Theorem

Suppose the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ exists and is continuous in an open interval containing a, and, furthermore, $f'(a) \neq 0$. Then, there exists some open interval, say I, such that $a \in I$, f is invertible on I, the inverse function f^{-1} has a derivative which is continuous on $f(I)$, and, furthermore,

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

Advanced Calculus Review / Primer

Inverse Function Theorem

Suppose $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable on an open set of $\mathbb{R}^n$ containing point $\mathbf{p}$. Suppose, furthermore, the Jacobian of F at $\mathbf{p}$,

$$J_F(\mathbf{p}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{p}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{p}) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{p}) & \cdots & \frac{\partial f_n}{\partial x_n}(\mathbf{p}) \end{bmatrix}$$ has a non-zero determinant.

Then, there exists some open set, say U, containing $\mathbf{p}$, such that F is invertible on U, the inverse function F^{-1} is continuously differentiable on $F(U)$, and, furthermore,

$$J_{F^{-1}}(F(\mathbf{p})) = J_F(\mathbf{p})^{-1}.$$

Proof See any advanced calculus text.

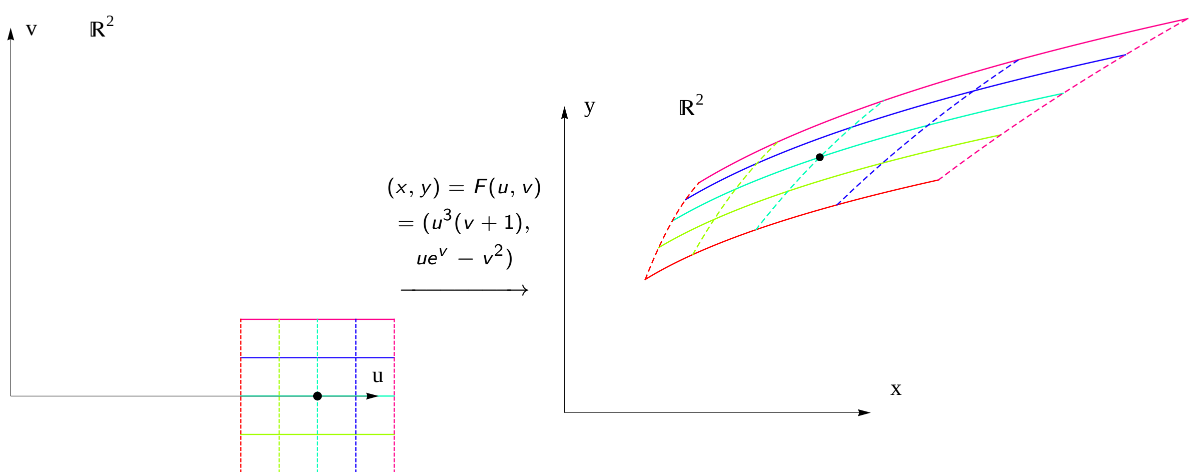
Advanced Calculus Review / Primer

Example

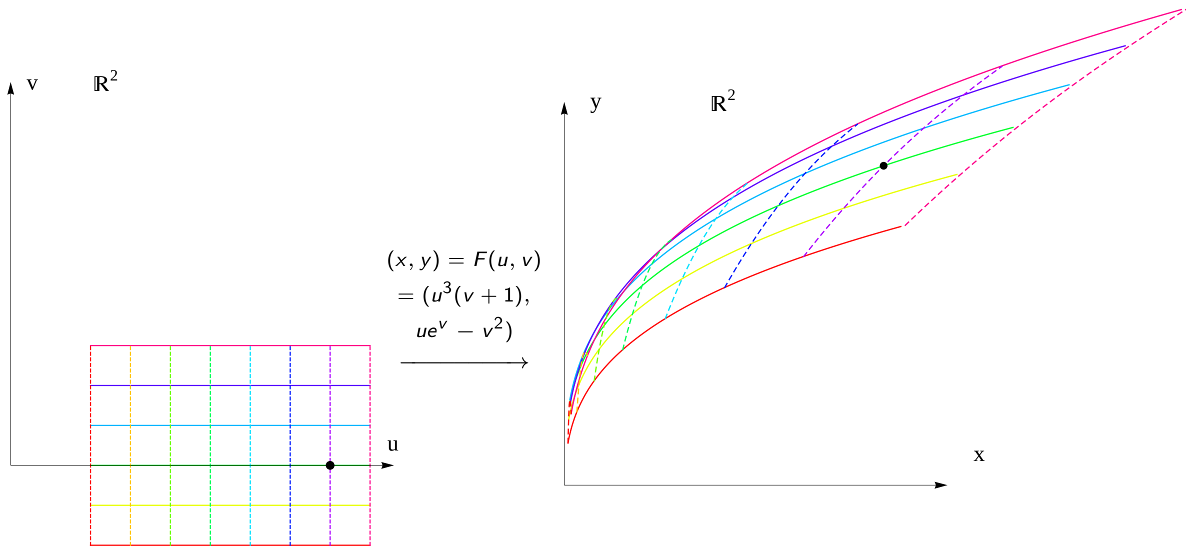
Let F be the map from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $F(u, v) = (u^3(v+1), ue^v - v^2)$, and let $\mathbf{p} = (1, 0)$.

$$J_F(\mathbf{p}) = \begin{bmatrix} 3u^2(v+1) & u^3 \\ e^v & ue^v - 2v \end{bmatrix}\Bigg|_{(1,0)} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

Since $|J_F(\mathbf{p})| = 2 \neq 0$, F must be differentiably invertible in a neighborhood of $\mathbf{p}$.



Advanced Calculus Review / Primer



Advanced Calculus Review / Primer

Calc I Theorem

Suppose the derivative of $g : \mathbb{R} \to \mathbb{R}$ exists at a, and the derivative of $f : \mathbb{R} \to \mathbb{R}$ exists at $g(a)$. Then the derivative of the composite function $f \circ g$ exists at a and

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Multidimensional generalization

Suppose $G : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{p}$, and $F : \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at $G(\mathbf{p})$. Then the derivative of the composite function $F \circ G$ exists at $\mathbf{p}$ and

$$J_{F \circ G}(\mathbf{p}) = J_F(G(\mathbf{p}))J_G(\mathbf{p}).$$

Example

$$G(u,v) = (\cos u, \sin u, v); \quad F(x,y,z) = (z, x^2); \quad \mathbf{p} = (\frac{\pi}{3}, 1)$$

$$J_G(\mathbf{p}) = \begin{bmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & 1 \end{bmatrix}\Big|_{(\frac{\pi}{3},1)} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$J_F(G(\mathbf{p})) = \begin{bmatrix} 0 & 0 & 1 \\ 2x & 0 & 0 \end{bmatrix}\Big|_{(\frac{1}{2},\frac{\sqrt{3}}{2},1)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J_F(G(\mathbf{p}))J_G(\mathbf{p}) = \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{3}}{2} & 0 \end{bmatrix}$$

Example (continued)

$$G(u,v) = (\cos u, \sin u, v); \quad F(x,y,z) = (z, x^2); \quad \mathbf{p} = (\frac{\pi}{3}, 1)$$

$$(F \circ G)(u,v) = F(\cos u, \sin u, v) = (v, \cos^2 u)$$

$$J_{(F\circ G)}(\mathbf{p}) = \begin{bmatrix} 0 & 1 \\ 2\cos u(-\sin u) & 0 \end{bmatrix}\Big|_{(\frac{\pi}{3},1)} = \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{3}}{2} & 0 \end{bmatrix}$$

So direct computation agrees with the chain rule computation. Of course.