4.1: LDEs (basic theory)

Recall that an IVP is an n^{th} order ODE
$$a_n(x)y^{(n)} + \cdots + a_0(x)y = g(x)$$ paired with side conditions of the form
$$y(x_0) = y_0,\quad y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

Thm Suppose $a_n, \ldots, a_0, g$ are continuous on an interval I and a_n is never 0 on I. Then if $x_0 \in I$, $\exists!$ soln on I to the IVP above.

Note: Any IVP with constant coeffs and a cont. forcing term $g(x)$ satisfies the conditions in the Thm.

A boundary value problem (BVP) pairs a LDE (of order 2 or higher) with side conditions involving more than one point.

The general 2^{nd} order (ordinary) BVP has the form
$$\begin{cases} a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \\ \alpha_1 y(a) + \beta_1 y'(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \end{cases}$$ where $a \neq b$.

There is no simple analog of Thm 4.1 for BVPs.

Ex: $y'' + y = 0$ is solved by the 2-parameter family
$$\boxed{y = c_1 \sin x + \cancel{c_2 \cos x}}$$

Ex: $$\begin{cases} y'' + y = 0 \\ y(0) = 0,\ y(\pi) = 0 \end{cases}$$ Infinitely many soln's: $y = c_1 \sin x$, c_1 anything!

$$\begin{cases} y'' + y = 0 \\ y(0) = 0,\ y\left(\frac{\pi}{2}\right) = 0 \end{cases}$$ Only the trivial soln.

$$\begin{cases} y'' + y = 0 \\ y(0) = 0,\ y(\pi) = 1 \end{cases}$$ No soln's!

Def 4.1 The set of fcns $\{f_1, \ldots, f_n\}$ is said to be linearly dependent on interval I if $\exists$ constants $c_1, \ldots, c_n$ (not all 0) such that
$$c_1 f_1(x) + \cdots + c_n f_n(x) \equiv 0 \text{ on } I.$$

Ex #1. $\{6x^2 - 2x + 4,\ 3x^2 + 2,\ x\}$ is linearly dependent on $(-\infty, \infty)$, since
$$(1)(6x^2 - 2x + 4) + (-2)(3x^2 + 2) + (2)(x) \equiv 0$$

Ex #2 $\{x, |x|\}$ is linearly dependent on $(-\infty, 0)$ and $(0, \infty)$, but linearly independent on $(-\infty, \infty)$.
$$(1)x + (1)|x| \equiv 0 \text{ for } x \in (-\infty, 0)$$
$$(1)x + (-1)|x| \equiv 0 \text{ for } x \in (0, \infty)$$

Suppose $c_1 x + c_2|x| \equiv 0$ for $x \in (-\infty, \infty)$

Then, in particular, $c_1(1) + c_2|1| = 0$
and $c_1(-1) + c_2|-1| = 0$.

Thus $$\begin{cases} c_1 + c_2 = 0 \\ -c_1 + c_2 = 0 \end{cases} \Rightarrow c_1 = c_2 = 0.$$

So $\{x, |x|\}$ is lin. ind on $(-\infty, \infty)$.

See you at 8:07 PM.