CHALLENGE PROBLEMS

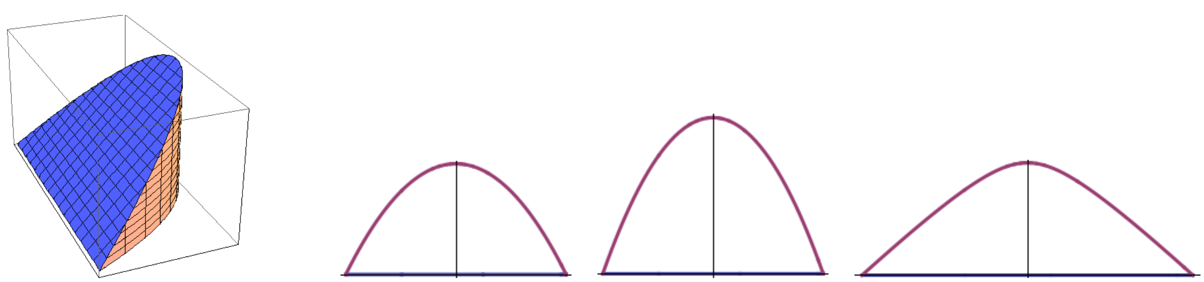
1. Determine whether or not $\displaystyle\lim_{(x,y)->(0,0)} \frac{x^{32}+x^5y^{10}}{x^{30}+y^{12}}$ exists.

2. Define f on $[-1,\infty)$ by $f(x)=\begin{cases} 0, & x=0 \\ \frac{1}{\lceil \frac{1}{x} \rceil}, & \text{otherwise} \end{cases}$. Determine the set of points D at which f is differentiable, and find f' everywhere on D. Justify your answer carefully.

3. Use MatLab (or other CAS of your choice) to write a function which randomly chooses a point on the unit sphere ($x^2+y^2+z^2=1$) according to the probability distribution which is uniform with respect to surface area. In other words for *any* region R on the unit sphere (with a well-defined area), the probability that a point is chosen from R should equal $\dfrac{\text{Area}(R)}{4\pi}$.

4. Show that $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$.

5. Let Q be the region of space above the x,y plane, below the plane $z=y$, and "left" of the parabolic cylinder $y=1-x^2$ (*i.e.* points in Q satisfy $y<1-x^2$). The boundary of Q breaks naturally into three pieces: a flat bottom, a flat top, and a curved side. Suppose you want to build a model using pieces cut from (bendable) card stock. Find equations for the boundary curves of the pieces (before bending).



(Alternatively design two pieces, one which will need to be folded, the other bent.)

6. Let $f:[-1,1]\to\mathbb{R}$ be given by $f(x)=\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^2}$. Determine the radius of convergence of the Taylor series for f centered at $\dfrac{-1}{2}$.

7. Prove that $e^{\pi}>\pi^{e}$.

8. Define f by $f(x)=\dfrac{1}{1+x^2}$. Find a simple, closed form representation for $f^{(n)}(x)$ which is valid for $n=1,2,3,\ldots$.

9. Suppose $f\in C_0^1(\mathbb{R}^+)$ and $\delta\in[0,1)$. Prove that

$$\int_0^{\infty}(f(t))^2t^{-\delta}dt \le \frac{4}{(1-\delta)^2}\int_0^{\infty}(f'(t))^2t^{2-\delta}dt.$$

(A function $f:[0,\infty)\to\mathbb{R}$ is in $C_0^1(\mathbb{R}^+)$ if (1) there exist a,b such that $0<a<b$ and $f(x)=0$ for all x **not** in $[a,b]$, and (2) f' exists and is continuous on $(0,\infty)$.) Thanks to Yueyue for contributing this problem from USTC (中科大).

10. Determine how many rectangles can be formed from the squares of a standard 8x8 chessboard. Generalize to an nxn board. Thanks to Hetian for contributing this problem.

11. Derive an exact expression for the angle between any two vectors which represent displacements from the center of a regular tetrahedron to two distinct vertices.

12. Prove $\pi \neq \frac{355}{113}$.

13. Determine whether or not $\sum_{n=1}^{\infty} \frac{(1)(3)\cdots(2n-1)}{2^n n!}$ converges. Use an elementary argument to prove your answer.