10.6

7) $\lim_{x\to \pi/2^-} \frac{\tan 5x}{\tan x} = \lim_{x\to\pi/2^-}\left[\left(\frac{\sin 5x}{\sin x}\right)\left(\frac{\cos x}{\cos 5x}\right)\right] = \frac{1}{5}$

since $\lim_{x\to\pi/2^-} \frac{\sin 5x}{\sin x} = 1$ and $\lim_{x\to\pi/2^-} \frac{\cos x}{\cos 5x} \overset{*}{=} \lim_{x\to\pi/2^-} \frac{\sin x}{5\sin 5x} = \frac{1}{5}$

20) Take log:

$\lim_{x\to\pi/2} \ln(|\sec x|^{\cos x}) = \lim_{x\to\pi/2} \cos x \ln|\sec x| = \lim_{x\to\pi/2} \frac{\ln|\sec x|}{\sec x}$

$\overset{*}{=} \lim_{x\to\pi/2} \frac{\tan x}{\sec x \tan x} = \lim_{x\to\pi/2} \cos x = 0$, so

$\lim_{x\to\pi/2} |\sec x|^{\cos x} = e^0 = 1$

21) $\lim_{x\to 0}\left[\frac{1}{\ln(1+x)} - \frac{1}{x}\right] = \lim_{x\to 0} \frac{x - \ln(1+x)}{x\ln(1+x)} \overset{*}{=} \lim_{x\to 0} \frac{x}{x + (1+x)\ln(1+x)}$

$\overset{*}{=} \lim_{x\to 0} \frac{1}{1+1+\ln(1+x)} = \frac{1}{2}$

30) Take log: $\lim_{x\to\infty} 3x\ln(1+\frac{1}{x}) = 3\lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} \overset{*}{=} 3\lim_{x\to\infty} \frac{\frac{-1/x^2}{1+1/x}}{\frac{-1}{x^2}} = 3$

so $\lim_{x\to\infty}(1+\frac{1}{x})^{3x} = e^3$

37) 1 : $\ln\left[(\ln n)^{1/n}\right] = \frac{1}{n}\ln(\ln n) \to 0$

41) 0 : $0 \le \frac{n^2 \ln n}{e^n} < \frac{n^3}{e^n}$, $\lim_{x\to\infty} \frac{x^3}{e^x} = 0$

53) $\frac{b}{a}\sqrt{x^2-a^2} - \frac{b}{a}x = \frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}+x}\left(\frac{b}{a}\right)\left(\sqrt{x^2-a^2}-x\right)$

$= \frac{-ab}{\sqrt{x^2-a^2}+x} \to 0$ as $x\to\infty$

57) $\lim_{x\to 0^-} \frac{-2x}{\cos x} \neq \lim_{x\to 0^-} \frac{2}{-\sin x}$

L'Hospital's rule does not apply here since $\lim_{x\to 0^-} \cos x = 1$

59) a) Let S be the set of positive integers for which the statement is true. Since $\lim_{x\to\infty} \frac{\ln x}{x} = 0$, $1 \in S$.

Assume that $k \in S$. By L'Hospital's rule,

$$\lim_{x\to\infty} \frac{(\ln x)^{k+1}}{x} \overset{*}{=} \lim_{x\to\infty} \frac{(k+1)(\ln x)^k}{x} = 0 \quad (\text{since } k\in S)$$

Thus $k+1 \in S$ and S is the set of positive integers.

b) Choose any positive number α. Let $k-1$ and k be positive integers such that $k-1 \le \alpha \le k$. Then, for $x > e$,

$$\frac{(\ln x)^{k-1}}{x} \le \frac{(\ln x)^{\alpha}}{x} \le \frac{(\ln x)^{k}}{x}$$

and the result follows by the pinching theorem

62) a) $\lim_{k\to 0^+} v(t) = \lim_{k\to 0^+} \frac{mg(1-e^{-(k/m)t})}{k} \overset{*}{=} \lim_{k\to 0^+} \frac{gte^{-(k/m)t}}{1} = gt$

b) $\frac{dv}{dt} = gt \Rightarrow v(t) = gt + C$; $v(0) = 0 \Rightarrow C = 0$

and $v(t) = gt$.

10.7

13) diverges: $\int_e^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_e^b \frac{\ln x}{x}\,dx =$

$$= \lim_{b\to\infty} \left[\frac{1}{2}(\ln x)^2\right]_e^b = \lim_{b\to\infty} \left[\frac{1}{2}(\ln(b))^2 - \frac{1}{2}\right] \to \infty$$

20) 2: $\int_{1/3}^{3} \frac{dx}{\sqrt[3]{3x-1}} = \lim_{a\to\frac{1}{3}^+} \int_a^3 \frac{dx}{(3x-1)^{1/3}} = \lim_{a\to\frac{1}{3}^+} \left[\frac{3(3x-1)^{2/3}}{2\cdot 3}\right]_a^3$

$$= \lim_{a\to\frac{1}{3}^+} \left[\frac{8^{2/3}}{2} - \frac{(3a-1)^{2/3}}{2}\right] = 2$$

23) 4: $\int_3^5 \frac{x}{\sqrt{x^2-9}}\,dx = \lim_{a\to 3^-} \int_a^5 x(x^2-9)^{-1/2}\,dx$

$$= \lim_{a\to 3^-} \left[(x^2-9)^{1/2}\right]_a^5 = \lim_{a\to 3} \left[4 - (a^2-9)^{1/2}\right] = 4$$

24) $\int_1^4 \frac{dx}{x^2-4} = \lim_{b\to 2^-} \int_1^b \frac{dx}{x^2-4} + \lim_{a\to 2^+} \int_a^4 \frac{dx}{x^2-4} =$

$$= \lim_{b\to 2^-} \left[\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|\right]_1^b + \lim_{a\to 2^+} \left[\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|\right]_a^4$$

$$= \lim_{b\to 2^-}\left(\frac{1}{4}\ln\left|\frac{b-2}{b+2}\right| - \frac{1}{4}\ln\frac{1}{3}\right) + \lim_{a\to 2^+}\left(\frac{1}{4}\ln\frac{2}{6} - \frac{1}{4}\ln\left|\frac{a-2}{a+2}\right|\right) \to \infty$$

diverges

34) 2: $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{a\to 0^+}\int_a^{\pi/2}\frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{a\to 0^+}\left[2\sqrt{\sin x}\right]_a^{\pi/2} = 2$

37) $\int_0^1 \sin^{-1}x\,dx = \left[x\sin^{-1}x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} - \lim_{a\to 1^-}\int_0^a \frac{x}{\sqrt{1-x^2}}\,dx$

(by parts)

Now, $\int_0^a \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_1^{1-a^2}\frac{1}{\sqrt{u}}\,du = \left[-\sqrt{u}\right]_1^{1-a^2} = 1-\sqrt{1-a^2}$

$(u = 1-x^2)$

Thus, $\int_0^1 \sin^{-1}x\,dx = \pi/2 - \lim_{a\to 1^-}\left(1-\sqrt{1-a^2}\right) = \pi/2 - 1$

38) a) For any r, we can find k such that $x^r < e^{x/2}$ for $x \geq k$
(since $e^{x/2}$ grows faster than any power of x.

Then $\int_0^\infty x^r e^{-x}\,dx < \int_0^k x^r e^{-x}\,dx + \int_k^\infty e^{-x/2}\,dx$ which converges.

Thus $\int_0^\infty x^r e^{-x}\,dx$ converges for all r.

b) For $n=1$: $\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_0^b = \lim_{b\to\infty}\left[-be^{-b} - e^{-b} + 1\right] = 1$

Assume true for n.

$$\int_0^\infty x^{n+1}e^{-x}\,dx = \lim_{b\to\infty}\left(\left[-x^{n+1}e^{-x}\right]_0^b + (n+1)\int_0^b x^n e^{-x}\,dx\right)$$

$$= \lim_{b\to\infty}\left(-b^{n+1}e^{-b}\right) + (n+1)\int_0^\infty x^n e^{-x}\,dx = 0 + (n+1)n! = (n+1)!$$

34) $\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \int_0^1 \frac{1}{\sqrt{x}(1+x)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$

$$= \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}(1+x)}\,dx + \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}(1+x)}\,dx$$

Now, $\int \frac{1}{\sqrt{x}(1+x)}\,dx = \int \frac{2}{1+u^2}\,du = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C$

$(u=\sqrt{x})$

Therefore, $\lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{a\to 0^+} \left[2\tan^{-1}\sqrt{x}\right]_a^1$

$$= \lim_{a\to 0^+} 2\left[\pi/4 - \tan^{-1}a\right] = \pi/2$$

and $\lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}(1+x)}\,dx = \lim_{b\to\infty} \left[2\tan^{-1}\sqrt{x}\right]_1^b = \lim_{b\to\infty} 2\left[\tan^{-1}b - \pi/4\right] = \pi/2$

Thus, $\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,dx = \pi$

56) Diverges ~~both~~ by comparison with $\int_e^\infty \frac{dx}{(x+1)\ln(x+1)}$

57) a) $\lim_{b\to\infty} \int_0^b \frac{2x}{1+x^2}\,dx = \lim_{b\to\infty} \left[\ln(1+x^2)\right]_0^b \to \infty$

Thus, the improper integral $\int_0^\infty \frac{2x}{1+x^2}\,dx$ diverges

b) $\lim_{b\to\infty} \int_{-b}^b \frac{2x}{1+x^2}\,dx = \lim_{b\to\infty} \left[\ln(1+x^2)\right]_{-b}^b$

$$= \lim_{b\to\infty} \left(\ln[1+b^2] - \ln[1+(-b)^2]\right) = \lim_{b\to\infty} (0) = 0$$

60) For all real t, $\frac{-t^2}{2} < t+1$. Therefore,

$\int_{-\infty}^{x} e^{-t^2/2}\, dt$ converges by comparison

with $\int_{-\infty}^{x} e^{t+1}\, dt$.