10.4

7) converges to 0: $\frac{x^{100n}}{n!} = \frac{(x^{100})^n}{n!} \to 0$

29) converges to 0: $0 < \int_n^{n+1} e^{-x^2}\,dx \le e^{-n^2}[(n+1)-n] = e^{-n^2} \to 0$

35) converges to 0: $\left| \int_{-1/n}^{1/n} \sin x^2\,dx \right| \le \int_{-1/n}^{1/n} |\sin x^2|\,dx \le \int_{-1/n}^{1/n} 1\,dx = \frac{2}{n} \to 0$

38) a) diverges: $\frac{n^2 \cos(n^2\pi)}{n^2+1}$ is close to 1 if n is even, -1 if n is odd

b) converges: $n(2^{1/n}-1) \to \ln 2$

c) converges: $\frac{n^e}{e^n} \to 0$

41) The length of each side of the polygon is $2r\sin(\pi/n)$. Therefore the perimiter, p_n, of the polygon is given by: $p_n = 2rn\sin(\pi/n)$

b) $2rn\sin(\pi/n) \to 2\pi r$ as $n \to \infty$: The number $2rn\sin(\pi/n)$ is the perimeter of a regular polygon of n sides inscribed in a circle of radius r. As n tends to ∞, the perimeter of the polygon tends to the circumference of the circle.

43) By the hint, $\lim_{n\to\infty} \frac{1+2+\dots+n}{n^2} = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{1+1/n}{2} = \frac{1}{2}$

44) diverges: $\frac{1^2+2^2+\dots+n^2}{(1+n)(2+n)} = \frac{n(n+1)(2n+1)}{6(1+n)(2+n)} = \frac{2n^3+3n^2+n}{6n^2+18n+12} \to \infty$

45) By the hint, $\lim_{n\to\infty} \frac{1^3+2^3+\dots+n^3}{2n^4+n-1} = \lim_{n\to\infty} \frac{n^2(n+1)^2}{4(2n^4+n-1)} = \lim_{n\to\infty} \frac{1+2/n+1/n^2}{8+4/n^3-4/n^4} = \frac{1}{8}$

47) a) $m_{n+1} - m_n = \frac{1}{n+1}(a_1+\dots+a_n+a_{n+1}) - \frac{1}{n}(a_1+\dots+a_n)$

$= \frac{1}{n(n+1)}\left[n a_{n+1} - (\underbrace{a_1+\dots+a_n}_{n})\right] > 0$ since $\{a_n\}$ is increasing

47b) we begin with the hint:

$$m_n < \frac{|a_1 + \dots + a_j|}{n} + \frac{\varepsilon}{2}\left(\frac{n-j}{n}\right).$$

since j is fixed,

$$\frac{|a_1 + \dots + a_j|}{n} \to 0$$

and therefore for n sufficiently large

$$\frac{|a_1 + \dots + a_j|}{n} < \frac{\varepsilon}{2}.$$

since $\frac{\varepsilon}{2}\left(\frac{n-j}{n}\right) < \frac{\varepsilon}{2}$ we see that, for n sufficiently large,

$|m_n| < \varepsilon$. This shows that $m_n \to 0$.

49) Let S be the set of positive integers n $(n \geq 2)$ for which the inequalities hold. Since

$$(\sqrt{b})^2 - 2\sqrt{ab} + (\sqrt{a})^2 = (\sqrt{b} - \sqrt{a})^2 > 0$$

it follows that $\frac{a+b}{2} > \sqrt{ab}$ and so $a_1 > b_1$. Now,

$$a_2 = \frac{a_1 + b_1}{2} < a_1 \quad \text{and} \quad b_2 = \sqrt{a_1 b_1} > b_1.$$

Also, by the argument above,

$$a_2 = \frac{a_1 + b_1}{2} > \sqrt{a_1 b_1} = b_2$$

and so $a_1 > a_2 > b_2 > b_1$. Thus, $2 \in S$. Assume that $k \in S$. Then

$$a_{k+1} = \frac{a_k + b_k}{2} < \frac{a_k + a_k}{2} = a_k, \quad b_{k+1} = \sqrt{a_k b_k} > \sqrt{b_k^2} = b_k$$

and

$$a_{k+1} = \frac{a_k + b_k}{2} > \sqrt{a_k b_k} = b_{k+1}.$$

Thus, $k+1 \in S$. Therefore, the inequalities hold for all $n \geq 2$.

49b) $\{a_n\}$ is a decreasing sequence which is bounded below.
$\{b_n\}$ is an increasing sequence which is bounded above.

Let $L_a = \lim\limits_{n\to\infty} a_n$, $L_b = \lim\limits_{n\to\infty} b_n$.

Then

$$a_n = \frac{a_{n-1} + b_{n-1}}{2} \Rightarrow L_a = \frac{L_a + L_b}{2} \text{ and } L_a = L_b .$$

<u>10.5</u>

1) $$\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} \overset{\star}{=} \lim_{x\to 0^+} 2\sqrt{x}\cos x = 0$$

25) $$\lim_{x\to \pi/2} \frac{\ln(\sin x)}{(\pi - 2x)^2} \overset{\star}{=} \lim_{x\to\pi/2} \frac{-\cot x}{4(\pi-2x)} \overset{\star}{=} \lim_{x\to\pi/2} \frac{\csc^2 x}{-8} = -\frac{1}{8}$$

45) The limit does not exist if $b \neq 1$. Therefore, $b=1$.

$$\lim_{x\to 0} \frac{\cos ax - 1}{2x^2} \overset{\star}{=} \lim_{x\to 0} \frac{-a\sin ax}{4x} \overset{\star}{=} \lim_{x\to 0} \frac{-a^2\cos ax}{4} = -\frac{a^2}{4} .$$

Now, $\frac{-a^2}{4} = -4 \Rightarrow a \pm 4$.

46) $$\lim_{x\to 0} \frac{\sin 2x + ax + bx^3}{x^3} \overset{\star}{=} \frac{2\cos 2x + a + 3bx^2}{3x^2}$$ need $a=-2$ to keep numerator 0.

$$\overset{\star}{=} \lim_{x\to 0} \frac{-4\sin 2x + 6bx}{6x} \overset{\star}{=} \lim_{x\to 0} \frac{-8\cos 2x + 6b}{6} = 0$$ if $6b = 8$

$$\Rightarrow a=-2 , b = \frac{4}{3}$$

48) $$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} \overset{\star}{=} \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)(-1)}{2} = f'(x)$$

(note that here we differentiated with respect to h, not x).

b) $$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \overset{\star}{=} \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

$$\overset{\star}{=} \lim_{h\to 0} \frac{f''(x+h) + f''(x-h)}{2} = f''(x)$$

50) $\lim\limits_{x\to 0} \dfrac{Si(x)}{x} \overset{\star}{=} \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

b) $\cancel{\lim\limits_{x\to 0} \sin x}$ $\lim\limits_{x\to 0} \dfrac{Si(x)-x}{x^3} \overset{\star}{=} \lim\limits_{x\to 0} \dfrac{Si(x)/x - 1}{3x^2} = \dfrac{\sin x - x}{3x^3}$

$\overset{\star}{=} \lim\limits_{x\to 0} \dfrac{\cos x - 1}{9x^2} \overset{\star}{=} \lim\limits_{x\to 0} \dfrac{-\sin x}{18x} = -\dfrac{1}{18}$

51) a) $\lim\limits_{x\to 0} \dfrac{C(x)}{x} \overset{\star}{=} \lim\limits_{x\to 0} \dfrac{\cos^2 x}{1} = 1$

b) $\lim\limits_{x\to 0} \dfrac{C(x)-x}{x^3} \overset{\star}{=} \lim\limits_{x\to 0} \dfrac{\cos^2 x - 1}{3x^2} \overset{\star}{=} \lim\limits_{x\to 0} \dfrac{-2\cos x \sin x}{6x} \overset{\star}{=} \lim\limits_{x\to 0} \dfrac{-2\cos^2 x + 2\sin^2 x}{6} = \dfrac{-1}{3}$

52) a) $\lim\limits_{x\to a} \dfrac{\int_a^x f(t)\,dt}{f(x)} \overset{\star}{=} \lim\limits_{x\to a} \dfrac{f(x)}{f'(x)} = 0$

b) similarly, $\lim\limits_{x\to a} \dfrac{\int_a^x f(t)\,dt}{f(x)} \overset{\star}{=} \lim\limits_{x\to a} \dfrac{f(x)}{f'(x)} \overset{\star}{=} \dots \overset{\star}{=} \dfrac{f^{(k-1)}(x)}{f^{(k)}(x)} = 0$.