11.8

3) Use the fact that $\frac{d^{(k-1)}}{dx^{(k-1)}}\left[\frac{1}{1-x}\right] = \frac{(k-1)!}{(1-x)^k}$:

$$\frac{1}{(1-x)^k} = \frac{1}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}\left[1+x+\dots+x^{k-1}+x^k+x^{k+1}+\dots+x^{n+k-1}+\dots\right]$$

$$= \frac{1}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}\left[x^{k-1}+x^k+x^{k+1}+\dots+x^{n+k-1}+\dots\right]$$

$$= 1 + kx + \frac{(k+1)k}{2}x^2 + \dots + \frac{(n+k-1)(n+k-2)\cdots(n+1)}{(k-1)!}x^n + \dots$$

$$= 1 + kx + \frac{(k+1)k}{2!}x^2 + \dots + \frac{(n+k-1)!}{n!(k-1)!}x^n + \dots$$

8) $\ln\cos x = -\int\frac{\sin x}{\cos x}dx = -\int \tan x\, dx = \frac{-x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17}{2520}x^8 - \dots + C$

$\ln\cos 0 = 0 \Rightarrow C = 0$

15) $\frac{2x}{1-x^2} = 2x\left(\frac{1}{1-x^2}\right) = 2x\sum_{k=0}^{\infty}(x^2)^k = \sum_{k=0}^{\infty} 2x^{2k+1}$

23) a) $\lim_{x\to 0}\frac{1-\cos x}{x^2} \overset{*}{=} \lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}$

b) $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\dots}{x^2} = \lim_{x\to 0}\left(\frac{1}{2}-\frac{x^2}{4!}+\frac{x^4}{6!}-\dots\right) = \frac{1}{2}$

25) a) $\lim_{x\to 0}\frac{\cos x-1}{x\sin x} \overset{*}{=} \lim_{x\to 0}\frac{-\sin x}{\sin x + x\cos x} \overset{*}{=} \lim_{x\to 0}\frac{-\cos x}{2\cos x - x\sin x} = -\frac{1}{2}$

b) $\lim_{x\to 0}\frac{\cos x-1}{x\sin x} = \frac{-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots}{x^2-\frac{x^4}{3!}+\frac{x^6}{5!}\dots} = \frac{-\frac{1}{2}+\frac{x^2}{4!}-\frac{x^4}{6!}+\dots}{1-\frac{x^2}{3!}+\frac{x^4}{5!}\dots} = -\frac{1}{2}$

27) $\int_0^x \frac{\ln(1+t)}{t}\,dt = \int_0^x \frac{1}{t}\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k\right)dt = \int_0^x \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^{k-1}\right)dt$

$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_0^x t^{k-1}\,dt = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} x^k, \quad -1 \le x \le 1$

29) $\int_0^x \frac{\tan^{-1} t}{t}\,dt = \int_0^x \frac{1}{t}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} t^{2k+1}\right)dt = \int_0^x \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} t^{2k}\right)dt$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \int_0^x t^{2k}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} x^{2k+1}, \quad -1 \le x \le 1$

31) $0.804 \le I \le 0.808$

$I = \int_0^1 \left(1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \dots\right)dx$

$= \left[x - \frac{x^4}{4} + \frac{x^7}{14} - \frac{x^{10}}{60} + \frac{x^{13}}{(13)(24)} - \dots\right]_0^1$

$= 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312} - \dots$

since $\frac{1}{312} < 0.01$ we can stop there:

$1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} \le I \le 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312}$ gives $0.804 \le I \le 0.808$

39) $I \approx 0.4485$ $\quad I = \int_0^{0.5} \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots\right)dx$

$= \left[x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \dots\right]_0^{1/2}$

$= \frac{1}{2} - \frac{1}{2^2 \cdot 2^2} + \frac{1}{3^2 \cdot 2^3} - \frac{1}{4^2 \cdot 2^4} + \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2\, 2^k}$

Now, $\frac{1}{8^2 \cdot 2^8} = \frac{1}{16{,}384} \approx 0.000061$ is the first term which is less than 0.0001. Thus

$\sum_{k=1}^{7} \frac{(-1)^{k-1}}{k^2 \cdot 2^k} < I < \sum_{k=1}^{8} \frac{(-1)^{k-1}}{k^2 \cdot 2^k} \qquad I \approx 0.4485$

41) e^{x^3}, by 11.5.5

43) $3x^2 e^{x^3} = \frac{d}{dx}(e^{x^3})$

44) a) $f(x) = \frac{e^x - 1}{x} = \frac{1}{x}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots$

b) $f'(x) = \frac{xe^x - e^x + 1}{x^2} = \frac{1}{2} + \frac{2x}{3!} + \frac{3x^2}{4!} + \dots + \frac{nx^{n-1}}{(n+1)!} + \dots$

$f'(1) = 1 = \sum_{k=1}^{\infty} \frac{k}{(k+1)!}$

49) If f is even, then the odd ordered ~~and~~ derivatives $f^{(2k-1)}$, $k = 1, 2, \dots$ are odd. This implies that ~~$\exists$~~ $f^{(2k-1)}(0) = 0$ for all k and so $a_{2k-1} = f^{(2k-1)}(0)/(2k-1)! = 0$ for all k

b) If f is odd, then all the even ordered derivatives f^{2k}, $k = 1, 2 \dots$ are odd. This implies that $f^{2k}(0) = 0$ for all k and so $a_{2k} = f^{2k}(0)/(2k)! = 0$ for all k

50) $f(0) = 1$, $f'(0) = -2f(0) = -2$, $f''(x) = -2f'(x) = 4f(x)$, $f''(0) = 4$

$f^{(n)}(x) = (-2)^n f(x)$, $f^{(n)}(0) = (-2)^n$, $f(x) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} x^k = \sum_{k=0}^{\infty} \frac{(-2x)^k}{k!} = e^{-2x}$

11.9

3) In (11.9.2), replace x by x^2 and take $\alpha = 1/2$ to obtain

$1 + \frac{1}{2}x^2 - \frac{1}{8}x^4$

11) a) $f(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$

In 11.9.2, replace x by x^2 and take $\alpha = -1/2$ to obtain

$$\frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k x^{2k}$$

By problem 2, this series has radius of convergence $r = 1$

b) $\sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt = \int_0^x \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k t^{2k}\,dt$

$= \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k \int_0^x t^{2k}\,dt = \sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{(-1)^k}{2k+1} x^{2k+1}$

By theorem 11.8.4, the radius of convergence is $r=1$

14) $\sqrt[5]{36} = (32+4)^{1/5} = 2\left(1+\frac{1}{8}\right)^{1/5} \approx 2\left[1+\frac{1}{5}\left(\frac{1}{8}\right) - \frac{4}{25}\cdot\frac{1}{2}\cdot\left(\frac{1}{8}\right)^2\right] \approx 2.0475$

17) $17^{-1/4} = (16+1)^{-1/4} = \frac{1}{2}\left(1+\frac{1}{16}\right)^{-1/4} \approx \frac{1}{2}\left[1 - \frac{1}{64} + \frac{5}{8192}\right] \approx 0.4925$