11.6

1) $f(x)=\sqrt{x} = x^{1/2}$ $\quad f(4)=2$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(4) = 1/4$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $\quad f''(4) = -1/32$

$f'''(x) = 3/8\, x^{-5/2}$ $\quad f'''(4) = 3/256$

$f^{(4)}(x) = -15/16\, x^{-7/2}$

$$P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

$$R_3(x) = \frac{f^{(4)}(c)}{4!}(x-4)^4 = -\frac{15}{16}\cdot\frac{1}{4!}c^{-7/2}(x-4)^4 = -\frac{5}{128c^{7/2}}(x-4)^4$$

where c is between 4 and x.

5) $f(x) = \tan^{-1}(x)$ $\quad f(1) = \pi/4$

$f'(x) = \frac{1}{1+x^2}$ $\quad f'(1) = 1/2$

$f''(x) = -2x/(1+x^2)^2$ $\quad f''(1) = -1/2$

$f'''(x) = 6x^2-2/(1+x^2)^2$ $\quad f'''(1) = \frac{1}{2}$

$f^{(4)}(x) = 24(x-x^3)/(1+x^2)^3$

$$P_3(x) = \pi/4 + \frac{1}{2}(x-1) - \frac{1/2}{2!}(x-1)^2 + \frac{1/2}{3!}(x-1)^3$$

$$= \pi/4 + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3$$

$$R_3(x) = \frac{f^{(4)}c}{4!}(x-1)^4 = \frac{24(c-c^3)}{(1+c^2)^4}\cdot\frac{1}{4!}(x-1)^4 = \frac{c-c^3}{(1+c^2)^4}(x-1)^4$$

where c is between 1 and x

9) $g(x) = -3 + 5(x+1) - 19(x+1)^2 + 20(x+1)^3 - 10(x+1)^4 + 2(x+1)^5$

$(-\infty, \infty)$

12) $\frac{1}{b+x} = \frac{1}{b+a+x-a} = \frac{1}{b+a} \cdot \frac{1}{1+\frac{x-a}{b+a}}$

$$= \frac{1}{b+a} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x-a}{b+a}\right)^k = \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{a+b}\right)^{k+1} (x-a)^k$$

$$(a-|a+b|,\ a+|a+b|)$$

26) $\ln(x^2) = 2\ln x = 2\ln(1+(x-1)) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k$

28) $\sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2} = \frac{1}{2} + \frac{1}{2}\cos 2(x-\pi/2) =$

$$= \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} 2^{2k} (x-\pi/2)^{2k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1}}{(2k)!} (x-\pi/2)^{2k}$$

32) $(x-1)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} x^k (-1)^{n-k}$

34) a) $\sin x = \sin a + (x-a)\cos a - \frac{(x-a)^2}{2!}\sin a + \frac{(x-a)^3}{3!}\cos a + \dots$

$$\cos x = \cos a - (x-a)\sin a - \frac{(x-a)^2}{2!}\cos a + \frac{(x-a)^3}{3!}\sin a + \dots$$

b) in both instances

$$\sum_{k=0}^{\infty} |a_k| \leq \sum_{k=0}^{\infty} \frac{|x-a|^k}{k!}$$

c) $\sin(x_1+x_2) = \sin x_1 + x_2\cos x_1 - \frac{(x_2)^2}{2!}\sin x_1 - \frac{(x_2)^3}{3!}\cos x_1 + \dots$

$$= \left(\sin x_1 - \frac{x_2^2}{2!}\sin x_1 + \frac{x_2^4}{4!}\sin x_1 - \dots\right) + \left(x_2\cos x_1 - \frac{x_2^3}{3!}\cos x_1 + \frac{x_2^5}{5!}\cos x_1 - \dots\right)$$

$$= \sin x_1\left(1 - \frac{x_2^2}{2!} + \frac{x_2^4}{4!} - \dots\right) + \cos x_1\left(x_2 - \frac{x_2^3}{3!} + \frac{x_2^5}{5!} - \dots\right)$$

$$= \sin x_1 \cos x_2 + \cos x_1 \sin x_2$$

The other formula can be derived in a similar manner.

11.7

1) a) converges b) absolutely converges c) ? d) diverges

2) a) absolutely converges b) diverges c) ? d) converges

7) converges only at 0; divergence test: $(-k)^{2k}x^{2k} \to 0$ only if $x=0$ and series clearly converges at $x=0$.

11) converges only at 0; divergence test: $\left(\frac{k}{100}\right)^k x^k \to 0$ only if $x=0$ and series clearly converges at $x=0$

25) Converges only at $x=1$; ratio test: $\frac{b_{k+1}}{b_k} = \frac{k^3}{(k+1)^2}|x-1| \to \infty$ if $x \neq 1$. The series clearly converges at $x=1$; otherwise it diverges.

31) $(-1,1)$; root test: $(b_k)^{1/k} = \left(1+\frac{1}{k}\right)|x| \to |x|$, series converges for $|x|<1$. At the endpoints $x=1$ and $x=-1$, the series diverges since there $b_k \not\to 0$ [recall $(1+1/k)^k \to e$

39) $\frac{3x^2}{4} + \frac{9x^4}{9} + \frac{27x^6}{16} + \frac{81x^8}{25} + \dots = \sum_{k=1}^{\infty} \frac{3^k}{(k+1)^2} x^{2k}$

$\left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$; ratio test: $\frac{b_{k+1}}{b_k} = \frac{3(k+1)^2}{(k+2)^2}x^2 \to 3x^2$,

series converges for $x^2 < 1/3$. At $x = \pm\frac{1}{\sqrt{3}}$, the series becomes $\sum \frac{1}{(k+1)^2} \approx \sum \frac{1}{n^2}$, a convergent series

p-series.

41) $\sum a_k (x-1)^k$ convergent at $x=3 \Rightarrow \sum a_k (x-1)^k$ is absolutely convergent on $(-1,3)$

a) $\sum a_k = \sum a_k (2-1)^k$; absolutely convergent

b) $\sum (-1)^k (a_k) = \sum a_k (0-1)^k$; absolutely convergent

c) $\sum (-1)^k a_k 2^k = \sum a_k (-1-1)^k$; ??

45) $\sum_{k=0}^{\infty} a_0 + a_1 x + a_2 x^2 + a_0 x^3 + a_1 x^4 + a_2 x^5 + a_0 x^6 + \ldots$

$$= \sum_{k=0}^{\infty} (a_0 + a_1 x + a_2 x^2) x^{3k}$$

$$= \sum_{k=0}^{\infty} (a_0 + a_1 x + a_2 x^2)(x^3)^k$$

48) By ratio test : $\left|\frac{a_{k+1}}{a_k}\right| \cdot |x| \to \frac{|x|}{r}$, so $\left|\frac{a_{k+1} x^{2(k+1)}}{a_k x^{2k}}\right| = \left|\frac{a_{k+1}}{a_k}\right| |x|^2 \to \frac{|x|^2}{r}$

$\Rightarrow$ radius of convergence is $\sqrt{r}$