11.4
3) diverges: \( \frac{k}{k+1} \to 1 \neq 0 \)

13) a) does not converge absolutely:
\[
\left( \sqrt[k+1]{k+1} - \sqrt[k]{k} \right) \cdot \frac{\sqrt[k+1]{k+1} + \sqrt[k]{k}}{\sqrt[k+1]{k+1} - \sqrt[k]{k}} = \frac{1}{\sqrt[k+1]{k+1} + \sqrt[k]{k}}
\]
and
\[
\sum \frac{1}{\sqrt[k+1]{k+1} + \sqrt[k]{k}} > \sum \frac{1}{2k+1} = \frac{1}{2} \sum \frac{1}{k} \quad (p\text{-series with } p<1)
\]
b) converges conditionally: Theorem 11.4.3

24) diverges: \( \left| \frac{a_{k+1}}{a_{k}} \right| = \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \left( \frac{k+1}{k} \right)^k > 1 \) so \( a_k \not\to 0 \)

26) a) \( \sum \frac{\cos \frac{k\pi}{k}}{k} = \sum \frac{(-1)^k}{k} \) converges does not converge absolutely
b) converges conditionally: Theorem 11.4.3

31) diverges: \( a_k \not\to 0 \)

36) error \( a_{n+1} = \frac{1}{10^{n+1}} \)
a) \( \frac{1}{10^{n+1}} < 10^{-3} \Rightarrow n \geq 3 \)
b) \( \frac{1}{10^{n+1}} < 10^{-4} \Rightarrow n \geq 4 \)

44) Yes. This can be shown by making slight changes in the proof of Theorem 11.4.3. The even partial sums \( S_{2m} \) are now nonnegative. Since \( S_{2m+2} \leq S_{2m} \), the sequence converges; say \( S_{2m} \to l \). Since \( S_{2m+1} = S_{2m} - a_{2m+1} \) and \( a_{2m+1} \to 0 \), we have \( S_{2m+1} \to l \). Thus, \( S_n \to l \).
47) a) since \( \sum |a_k| \) converges, \( \sum |a_k|^2 = \sum a_k^2 \) converges

b) \( \sum \frac{1}{k^2} \) converges, \( \sum (-1)^{k+1} \frac{1}{k} \) is not absolutely convergent

50) a) \( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(a+b) + (a-b)}{2k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(a+b)}{2k} + \sum_{k=1}^{\infty} \frac{a-b}{2k} \)

b) The series is absolutely convergent if \( a=b=0 \);
    conditionally convergent if \( a=b \neq 0 \); divergent if \( a \neq b \).

\[ 11.5 \]

8) \( x - \frac{1}{2}x^5 \)

21) \[ |f(y) - P_n(y)| = |R_n(y)| \leq (3)^{(12)^{m!}} \leq \frac{3}{2^{m!(12!)}} \]

The least integer \( n \) that satisfies the inequality

\( \frac{3}{2^{m!(12!)}} < 0.000005 \) is \( n=9 \)

25) The Taylor polynomial

\( P_n(0.5) = 1 + (0.5)^2 + \frac{(0.5)^2}{2!} + \ldots + \frac{(0.5)^n}{n!} \)

estimates \( e^{0.5} \) within

\( |R_{n+1}(0.5)| \leq e^{0.5} \frac{10.5^{n+1}}{(n+1)!} < 2 \frac{0.5^{n+1}}{(n+1)!} \),

Since \( 2 \frac{(0.5)^9}{9!} = 0.00824 < 0.01 \), we can take \( n=3 \)

and be sure that

\( P_3(0.5) = 1 + (0.5)^2 + \frac{(0.5)^2}{2} + \frac{(0.5)^3}{6} = \frac{79}{48} \),

This differs from \( \sqrt{e} \) by less than 0.01.

Our calculator gives

\( \sqrt{e} \approx 1.64583 \quad \sqrt{e} \approx 1.6487213 \)
26) At \( x = 1.2 \) the logarithm series (11.58) gives
\[
\ln(1.2) = \ln(1 + 0.2) = 0.2 - \frac{1}{2}(0.2)^2 + \frac{1}{3}(0.2)^3 - \ldots
\]
This is a convergent alternating series with decreasing terms. The first term of magnitude less than 0.01 is \((0.2)^3/3 \approx 0.00767\).
Thus, \(0.2 - \frac{1}{2}(0.2)^2 = 0.18\) differs from \(\ln(1.2)\) by less than 0.01.
Our calculator gives \(\ln(1.2) \approx 0.182321\).

32) At \( x = 6^\circ = \frac{\pi}{30}\), the cosine series gives
\[
\cos \left(\frac{\pi}{30}\right) = 1 - \frac{1}{2}(\frac{\pi}{30})^2 + \frac{1}{4!}(\frac{\pi}{30})^4 - \frac{1}{6!}(\frac{\pi}{30})^6 + \ldots
\]
The first term less than 0.01 is \(\frac{1}{2}(\frac{\pi}{30})^2 \approx 0.0055\), so \(1\) differs from \(\cos 6^\circ\) by less than 0.01. Our calculator gives \(\cos 6^\circ \approx 0.9945219\).

43) \( f(x) = \frac{1}{1-x} \), \( f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}} \), \( k = 0, 1, 2, \ldots \)
\[
P_n(x) = \frac{(x+1)^{n+1}}{(1-x)^{n+1}} = \frac{1}{(1-c)^{n+2}} x^{n+1} \]
where \( c \) is between 0 and \( x \).

45) By (11.58) \( p_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1} \frac{x^n}{n} \)
For \( 0 \leq x \leq 1 \) we know from (11.45) that
\[
|P_n(x) - \ln(1+x)| < \frac{x^{n+1}}{n+1}
\]
a) \( n = 4 : \frac{(0.5)^{n+1}}{n+1} \leq 0.01 \Rightarrow 100 \leq (n+1) \cdot 2^{n+1} \Rightarrow n \geq 4 \)

b) \( n = 2 : \frac{(0.3)^{n+1}}{n+1} \leq 0.01 \Rightarrow 100 \leq (n+1) \cdot (\frac{10}{3})^{n+1} \Rightarrow n \geq 2 \)

c) \( n = 999 : \frac{(0.1)^{n+1}}{n+1} \leq 0.001 \Rightarrow 1000 \leq n+1 \Rightarrow n \geq 999 \)

49) The result follows from the fact that

\[
P^{(k)}(0) > \begin{cases} \frac{k! a_k}{n^k} & 0 \leq k \leq n \\ 0 & n < k \end{cases}
\]

50) Straightforward

52) \( \cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \) because the odd terms cancel

58) \( f(x) = \ln \left( \frac{1+x}{1-x} \right) = \ln (1+x) - \ln (1-x) \); \( f(0) = 0 \)

\[
f'(x) = \frac{1}{1+x} + \frac{1}{1-x} \quad f'(0) = 2
\]

\[
f''(x) = \frac{-1}{(1+x)^2} + \frac{1}{(1-x)^2} \quad f''(0) = 0
\]

\[
f'''(x) = \frac{2}{(1+x)^3} + \frac{2}{(1-x)^3} \quad f'''(0) = 4
\]

In general, \( f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n} + \frac{(n-1)!}{(1-x)^n} \); \( f^{(n)}(0) = 2(n-1)! \)

for \( n \) odd, \( 0 \) for \( n \) even. The result follows
b) Let \( g(x) = \frac{x^n}{e^{\frac{x^2}{2}}} \). Then \( \lim_{x \to 0} g(x) \) has form \( \frac{\infty}{\infty} \).

Successive applications of L'Hopital's rule will finally produce a quotient of the form

\[
\frac{c x^k}{e^{\frac{x^2}{2}}} ,
\]

where \( k \) is a nonnegative integer and \( c \) is a constant. It follows that

\[
\lim_{x \to 0} g(x) = 0
\]

c) \( f'(0) = \lim_{x \to 0} \frac{e^{-\frac{x^2}{2}} - 0}{x} = 0 \) by part b. Assume that \( f'(0) = 0 \). Then

\[
f^{(k+1)}(0) = \lim_{x \to 0} \frac{f^{(k)}(x) - 0}{x} = \lim_{x \to 0} \frac{f^{(k)}(0)}{x},
\]

Now, \( \frac{f^{(k)}(0)}{x} \) is a sum of terms of the form

\[
\frac{c e^{-\frac{x^2}{2}}}{x^n}, \quad n \text{ a positive integer and } c \text{ a constant}
\]

Again, by part b, \( f^{(k+1)}(0) = 0 \). Therefore,

\[
f^{(n)}(0) = 0 \text{ for all } n.
\]

d) \( 0 \quad \text{e) } x = 0 \)