HW #10

11.4

3) diverges: $\frac{k}{k+1} \to 1 \neq 0$

13) a) does not converge absolutely:

$$\left(\sqrt{k+1} - \sqrt{k}\right) \cdot \frac{\sqrt{k+1} + \sqrt{k}}{\sqrt{k+1} + \sqrt{k}} = \frac{1}{\sqrt{k+1} + \sqrt{k}}$$

and $\sum \frac{1}{\sqrt{k} + \sqrt{k+1}} > \sum \frac{1}{2\sqrt{k+1}} = \frac{1}{2} \sum \frac{1}{\sqrt{k+1}}$ (p-series with $p<1$)

b) converges conditionally: Theorem 11.4.3

24) diverges: $\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \left(\frac{k+1}{k}\right)^k > 1$ so $a_k \not\to 0$

26) a) $\sum \frac{\cos \pi k}{k} = \sum \frac{(-1)^k}{k}$ ~~converges~~ does not converge absolutely

b) converges conditionally: Theorem 11.4.3

31) diverges: $a_k \not\to 0$

36) error $< a_{n+1} = \frac{1}{10^{n+1}}$ a) $\frac{1}{10^{n+1}} < 10^{-3} \Rightarrow n \geq 3$

b) $\frac{1}{10^{n+1}} < 10^{-4} \Rightarrow n \geq 4$

44) Yes. This can be shown by making slight changes in the proof of Theorem 11.4.3. The even partial sums S_{2m} are now nonnegative. Since $S_{2m+2} \leq S_{2m}$, the sequence converges; say $S_{2m} \to l$. Since $S_{2m+1} = S_{2m} - a_{2m+1}$ and $a_{2m+1} \to 0$, we have $S_{2m+1} \to l$. Thus, $S_n \to l$.

47) a) Since $\sum |a_k|$ converges, $\sum |a_k|^2 = \sum a_k^2$ converges

b) $\sum \frac{1}{k^2}$ converges, $\sum (-1)^k \frac{1}{k}$ is not absolutely convergent

50) a) $$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(a+b)+(a-b)}{2k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(a+b)}{2k} + \sum_{k=1}^{\infty} \frac{a-b}{2k}$$

b) The series is absolutely convergent if $a=b=0$; Conditionally convergent if $a=b\neq 0$; divergent if $a\neq b$.

11.5

8) $x - \frac{1}{2}x^5$

21) $$|f(1/2) - P_n(1/2)| = |R_n(1/2)| \leq (3)\frac{(1/2)^{n+1}}{(n+1)!} = \frac{3}{2^{n+1}(n+1)!}$$

The least integer n that satisfies the inequality $\frac{3}{2^{n+1}(n+1)!} < 0.00005$ is $n=9$

25) The Taylor polynomial
$$P_n(0.5) = 1 + (0.5) + \frac{(0.5)^2}{2!} + \ldots + \frac{(0.5)^n}{n!}$$

estimates $e^{0.5}$ within

$$|R_{n+1}(0.5)| \leq e^{0.5}\frac{|0.5|^{n+1}}{(n+1)!} < 2\frac{(0.5)^{n+1}}{(n+1)!}.$$

Since $2\frac{(0.5)^4}{4!} = \frac{1}{8(24)} < 0.01$, we can take $n=3$ and be sure that
$$P_3(0.5) = 1 + (0.5) + \frac{(0.5)^2}{2} + \frac{(0.5)^3}{6} = \frac{79}{48}.$$
This differs from $\sqrt{e}$ by less than 0.01.
Our calculator gives
$$\frac{79}{48} \approx 1.645833 \quad \sqrt{e} \approx 1.6487213$$

28) At $x=1.2$ the logarithm series (11.5.8) gives

$$\ln(1.2) = \ln(1+0.2) = 0.2 - \tfrac{1}{2}(0.2)^2 + \tfrac{1}{3}(0.2)^3 - \ldots$$

This is a convergent alternating series with decreasing terms. The first term of magnitude less than 0.01 is $(0.2)^3/3 \approx 0.00267$.

Thus,

$$0.2 - \tfrac{1}{2}(0.2)^2 = 0.18$$

differs from $\ln(1.2)$ by less than 0.01.

Our calculator gives $\ln(1.2) \approx 0.1823215$

32) At $x = 6^\circ = \pi/30$, the cosine series gives

$$\cos \pi/30 = 1 - \tfrac{1}{2}(\pi/30)^2 + \tfrac{1}{4!}(\pi/30)^4 - \tfrac{1}{6!}(\pi/30)^6 + \ldots$$

The first term less than 0.01 is

$\frac{1}{2}(\pi/30)^2 \approx 0.0055$, so 1 differs from $\cos 6^\circ$

by less than 0.01. Our calculator gives

$\cos 6^\circ \approx 0.9945219$

43) $f(x) = \frac{1}{1-x}$, $f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$, $k = 0, 1, 2 \ldots$

$$R_n(x) = \frac{(n+1)!}{(1-c)^{n+2}(n+1)!} x^{n+1} = \frac{1}{(1-c)^{n+2}} x^{n+1}$$ where

c is between 0 and x.

45) By (11.5.8)

$$P_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n}$$

For $0 \le x \le 1$ we know from (11.4.5) that

$$|P_n(x) - \ln(1+x)| < \frac{x^{n+1}}{n+1}$$

a) $n=4$: $\frac{(0.5)^{n+1}}{n+1} \le 0.01 \Rightarrow 100 \le (n+1)\cdot 2^{n+1} \Rightarrow n \ge 4$

b) $n=2$; $\frac{(0.3)^{n+1}}{n+1} \le 0.01 \Rightarrow 100 \le (n+1)\left(\frac{10}{3}\right)^{n+1} \Rightarrow n \ge 2$

c) $n=999$: $\frac{(1)^{n+1}}{n+1} \le 0.001 \Rightarrow 1000 \le n+1 \Rightarrow n \ge 999$

49) The result follows from the fact that

$$p^{(k)}(0) = \begin{cases} k!\,a_k & 0 \le k \le n \\ 0 & n < k \end{cases}$$

50) Straightforward

52) $\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$

$= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ because the odd terms cancel

58) $f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$; $f(0)=0$

$f'(x) = \frac{1}{1+x} + \frac{1}{1-x}$ $\quad f'(0) = 2$

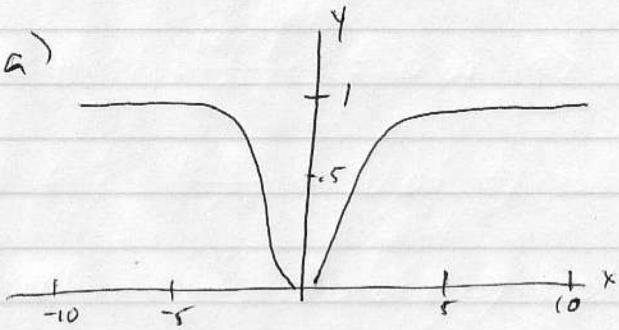
$f''(x) = \frac{-1}{(1+x)^2} + \frac{1}{(1-x)^2}$ $\quad f''(0) = 0$

$f'''(x) = \frac{2}{(1+x)^3} + \frac{2}{(1-x)^3}$ $\quad f'''(0) = 4$

In general, $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n} + \frac{(n-1)!}{(1-x)^n}$, $f^n(0) = 2(n-1)!$

for n odd, 0 for n even. The result follows

63) a)



b) Let $g(x) = \dfrac{x^{-n}}{e^{1/x^2}}$. Then $\lim\limits_{x \to 0} g(x)$ has form $\frac{\infty}{\infty}$.

Successive applications of L'Hospital's rule will finally produce a quotient of the form

$\dfrac{cx^k}{e^{1/x^2}}$, where k is a nonnegative integer and c is a constant. It follows that

$$\lim_{x \to 0} g(x) = 0$$

c) $f'(0) = \lim\limits_{x \to 0} \dfrac{e^{-1/x^2} - 0}{x} = 0$ by part b. Assume

that $f^{(k)}(0) = 0$. Then

$$f^{(k+1)}(0) = \lim_{x \to 0} \frac{f^{(k)}(x) - 0}{x} = \lim_{x \to 0} \frac{f^{(k)}(x)}{x}.$$

Now, $\dfrac{f^{(k)}(x)}{x}$ is a sum of terms of the form

$\dfrac{ce^{-1/x^2}}{x^n}$, n a positive integer and c a constant.

Again, by part b, $f^{(k+1)}(0) = 0$. Therefore,

$f^{(n)}(0) = 0$ for all n.

d) 0 e) $x = 0$